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Filtering Nonuniformly Sampled Grid-Based Signals

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***Abstract*—This paper presents an example application of digital alias-free signal processing, where a sequence of irregularly spaced, yet uniformly gridded, samples of a bandlimited discrete-time signal is filtered by using an oversampled finite impulse response filter. The mathematical model of the proposed filter is introduced, and a new interpolation formula for calculating the convolution operation of the filter, based on nonuniform sampling, is derived. In addition, uniform grid versions of Total Random, Stratified and Antithetical Stratified random sampling techniques are demonstrated. We carry out numerical comparison between these techniques and the proposed one in terms of Fourier transform estimates of the filtered output signal. The proposed interpolation technique shows enhancements over other sampling techniques after certain number of sampling points. Furthermore, it has a faster uniform convergence rate of the normalized root mean squared error than other techniques.**

Keywords-digital alias-free signal processing; random sampling; FIR filter; nonuniform interpolation

I. INTRODUCTION

Continuous-time signals are either uniformly or nonuniformly sampled in order to be converted to digital form. In uniform sampling, signal amplitude is acquired at evenly spaced time instants. Whereas, irregular time samples are used in nonuniform sampling (NUS), which can be a result of intentional or unintentional reasons. Sometimes, we have no control over the time presence of a signal, and so, it can't be sampled regularly. Examples of such unintentional NUS are found in astronomy, medicine, wireless communications and hardware manufacturing imperfections [1]-[3]. However, we may deliberately sample signals nonuniformly, for example, to circumvent aliasing problems in conventional digital signal processing (DSP), to compress data and save memory or just to reduce computational cost. Applications include IT and computer networks, signal processing, filter design, Fourier transform (FT), wideband spectrum sensing and compressed sensing, cognitive radio and radar [4]-[6].

The notion of NUS of deterministic continuous-time signals to avoid aliasing has been addressed by some researchers in the last century [7]-[9]. But, conventionally, it was coined by Shapiro and Silverman [10] who showed that alias-free sampling could be performed with sampling frequency less than the Nyquist rate, f_{Nyq}. Bilinskis introduced in [11] digital alias-free signal processing (DASP), where new techniques, methods and algorithms were proposed in attempts to overcome the aliasing problem, or at least decrease its harmful effects, in conventional DSP.

Tarczynski and Najib have discussed in [12] the use of NUS to estimate the FT of irregularly sampled signals. Having introduced the total random (ToRa) sampling technique, they proved that the uniform convergence rate of $1/N_r$ can be achieved using ToRa, where N_r denotes the number of random sampling points. Another technique to estimate the FT, named stratified sampling (StSa), was presented by Masry [13], where the observation period is divided into strata (time slots), and one sample is taken randomly per each stratum. In StSa, reduced estimation errors have been achieved. Later in 2009, He published another paper [14] in which the antithetical stratified (AnSt) random sampling technique was used to estimate the FT. The new stratum in AnSt is designed to include two sampling points: the first one is taken randomly within the stratum (exactly as in StSa), while the other one is a mirror reflection of the first point with regards to the centre of the stratum. Improved results were achieved, but only if the signal being sampled has a continuous second order derivative and is monotonically smooth.

The above-mentioned techniques depend on the simple Rectangle rule to estimate the FT. Eventually, estimating a particular FT component means calculating the area under the curve of the product of signal NUS points and their associated complex exponentials. So, we propose a new nonuniform interpolation technique, CS3NS, based on Lagrange second-order polynomial [15], to estimate both the output of the convolution operation of a finite impulse response (FIR) filter and the FT of the filtered output signal. All estimated FTs are then compared to the FT of a reference signal.

NUS-based asynchronous filtering was discussed in [16]-[17], where level-crossing approach was used as the random sampling technique, but this is out of the scope of this paper.

The rest of the paper comprises four sections: uniform grid filter model, mathematical formulation of the convolution, the proposed CS3NS interpolation rule and numerical results. A conclusion is also provided at the end of the paper.

II. UNIFORM GRID FILTER MODEL

There are some challenges in randomized signal processing with regards to time synchronization. For example, doing mathematical operations between time-based random variables, or random and non-random variables, is very difficult since they are often not accurately aligned. Doing so could lead to large estimation errors or even incorrect results.

To overcome this challenge, and for the sake of potential practical implementation of our research, we assume that the samples of the input signal are taken “randomly” from otherwise uniformly oversampled grid-based signal. This is why we are going to build a bandpass FIR filter model using dense and equally spaced impulse response samples, so that there will always be an exact time-match between signal samples and filter samples. Therefore, the impulse response of the proposed FIR filter is uniformly oversampled with frequency $F_s \gg f_{Nyq}$, and has a grid time step of $T_s = 1/F_s$. This means signal samples are also integer-multiples of T_s, as shown in Fig. 1. Choosing F_s depends on the resources available to a given application. However, the higher the sampling rate the more accurate the output results. Moreover, we assume that the proposed filter model will be stored offline as a lookup table in the memory buffer of an application hardware [16]. This will reduce the computational cost of filtering as no real-time interpolation is to be used.

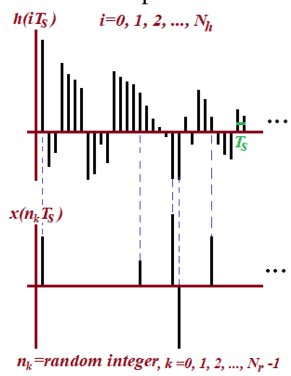

Fig. 1. Uniformly oversampled filter impulse response, $h(iT_s)$, timely synced with nonuniformly sampled grid-based input signal $x(n_kT_s)$.

III. Mathematical Formulation

A. Convolution of the Input Signal

We are considering an input continuous-time signal $x(t)$ that is densely and uniformly sampled in the time interval $[0, T]$, to produce the discrete-time version $x_u(kT_s)$. A reference signal, $y_u(iT_s)$, is defined as the FIR filtered output signal of $x_u(kT_s)$, where both k and i are integers, and $N_u = T/Ts$, is the total number of uniform samples:

$$y_u(iT_s) = T_s \sum_{k=0}^{N_u-1} x_u(kT_s)\, h(iT_s - kT_s). \qquad (1)$$

Our aim is to filter discrete-time nonuniformly sampled grid-based versions of $x(t)$ with the proposed FIR filter above, and compare the FTs of the filtered output signals with that of $y_u(iT_s)$. Note that $y_u(iT_s)$ is just the discrete form of the windowed convolution signal $y_T(t)$,

$$y_T(t) = \int_0^T x(\tau) h(t-\tau) d\tau. \qquad (2)$$

Fig. 2 shows an example of how those nonuniformly sampled versions of $x(t)$ are taken. More specifically, it introduces modified versions of ToRa, StSa and AnSt, where the sample time instants are accurately aligned with specific points on the uniform grid. The grid time instants are interpreted at $t_i = iT_s$, $i = 0, 1, 2, \dots, N_u$. Whereas the signal NUS points themselves occur at time instants $t_k = n_kT_s$, with

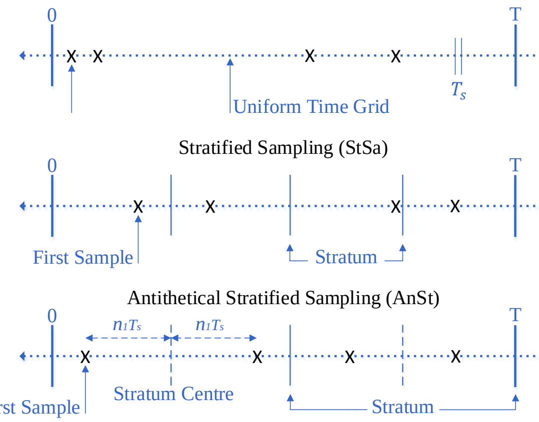

Fig. 2. Four-sample example of modified ToRa, StSa and AnSt random sampling techniques, where each signal sample exactly matches one of the uniform time grid instants.

$k = 0, 1, 2, \dots, N_r - 1$, and N_r is the number of random sampling points. Note that the arbitrary integers $n_k \in \{0, N_u\}$, and they depend on the selected random sampling technique.

B. Filter Convolution Based on Simple Rectangle Rule

For modified ToRa random sampling scheme and $[0, T]$ observation time window, we are considering a total of N_r i.i.d. arbitrary samples of a grid-based discrete-time input signal $x_\Delta(t_k)$. The probability density function (PDF) of t_k is $p_{t_k}(t_k) = 1/N_r$ for $t_k \in \{0, 1/T_s, 2/T_s, \dots, T\}$ and zero elsewhere. So, the nonuniform filtered output discrete-time signal, $y_\Delta(t_j)$, using the dense uniform grid filter, $h(iT_s)$, is

$$y_\Delta(t_j) = \sum_{k=0}^{N_r-1} x_\Delta(t_k)\, h(t_j - t_k)\, \Delta t_k, \qquad (3)$$

where t_j are unequally spaced time instants, but they are also integer multiples of T_s, i.e. $t_j = n_jT_s$. Moreover, Δt_k is the time difference between two consecutive samples, and so, $\Delta t_k = n_{k+1}T_s - n_kT_s = T_s\Delta n_k$, where $\Delta n_k = n_{k+1} - n_k$. So

$$y_\Delta(n_jT_s) = \sum_{k=0}^{N_r-1} x_\Delta(n_kT_s)\, h(n_jT_s - n_kT_s)\, \Delta n_kT_s. \qquad (4)$$

Note that Δn_k has the same PDF as t_k, i.e. $1/N_r$. Hence, the summand in (4) is a product of N_r components that all are random variables and have the same PDF. Therefore, the expected value of the estimator in (4) can be calculated by adding up the individual expected values of all components of the summation. For each component, we have

$$E\{y_\Delta^c(n_jT_s)\} = E\{x_\Delta(n_kT_s)\, h(n_jT_s - n_kT_s)\, \Delta n_kT_s\} \qquad (5a)$$

$$= \int_{-\infty}^{\infty} x_\Delta(n_kT_s)\, h(n_jT_s - n_kT_s) p_{t_k}(t_k)\, dn_kT_s. \qquad (5b)$$

However, $p_{t_k}(t_k)$ equals 0 outside $[0, T]$, so (5b) becomes

$$= \frac{1}{N_r} \int_0^T x_\Delta(n_kT_s)\, h(n_jT_s - n_kT_s)\, dn_kT_s. \qquad (5c)$$

$$E\{y_\Delta^c(n_jT_s)\} = \frac{1}{N_r} y_T(t). \qquad (5d)$$

For N_r components of $y_\Delta^c(n_jT_s)$, the expectation is

$$E\{y_\Delta(n_jT_s)\} = N_r \frac{1}{N_r} y_T(t) = y_T(t), \qquad (5e)$$

which means that the estimator in (4) is unbiased. Now the question arises about the quality of estimation, where the mean squared error (MSE) could be a good metric in this case, but since the estimator in (4) is unbiased, the MSE is the same as the variance. Thus,

$$\sigma_{y_\Delta}{}^2 = E\left\{\left|y_\Delta(n_j T_s)\right|^2\right\} - \left|E\{y_\Delta(n_j T_s)\}\right|^2. \quad (6)$$

Working out the calculations in (6), we find the variance as

$$\sigma_{y_\Delta}{}^2 = \left(T\mathrm{E}_{y_T} - y_T{}^2(t)\right)/N_r, \quad (7)$$

where E_{y_T} is the total energy of y_T in the interval $[0, T]$. This means that the quality of filtering estimation is proportional to $N_r{}^{-1}$ for ToRa, which coincides with the quality of FT estimation for ToRa in [12]. Same analyses can be carried out for the cases of StSa and AnSt, but will not be demonstrated here because of space limitation.

IV. The Proposed CS3NS Interpolation Rule

The convolution operation of the filter in (3) simply uses the Rectangle rule to estimate the filtered output signal, where the product $x_\Delta(t_k)\, h(t_j - t_k)$, representing the amplitude of a function, is multiplied by a given time width Δt_k and added accumulatively to calculate the total area under the curve (AUC) of the function within the specified interval. However, we can calculate the same area using other interpolation techniques. Hence, we propose here a grid-based nonuniform interpolation technique, where it is, up to our knowledge, not addressed in literature before. We call it composite Simpson 3-nonuniform-sample (CS3NS) rule, named after the traditional composite Simpson's 1/3 rule that is usually used in second-order polynomial interpolation of uniform samples.

A. Estimated Area

Suppose we have a continuous-time function, $f(t)$, and we need to estimate the area under $f(t)$ within the interval $[0, T]$ by interpolating specific number of nonuniformly spaced samples of it. Hence, we divide the whole interval $[0, T]$ into subintervals according to the number of sampling points. Every three consecutive samples constitute one subinterval, as shown in Fig. 3. Left and right samples (borders) of a given subinterval are shared with previous and next subintervals to form the composite rule and to calculate the total interpolated area from 0 to T.

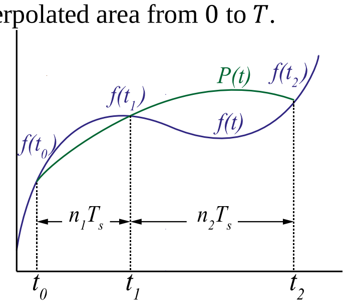

Fig. 3. One subinterval, $[t_0, t_2]$, of the proposed CS3NS rule.

Starting with the first three nonuniform samples, at time instants $t_0 = 0$, t_1 and t_2 (one subinterval), and recalling that time spacing between any two samples is integer multiple of T_s, we introduce the integer numbers n_1 and n_2 as follows: $t_1 - t_0 = n_1 T_s$ and $t_2 - t_1 = n_2 T_s$. Now, we estimate the subinterval area under $f(t)$ from t_0 to $t_2 = (n_1 + n_2)T_s$ by simply interpolating the three sample points $f(t_0)$, $f(t_1)$ and $f(t_2)$ using a second-order Lagrange polynomial $P(t) = a_0 + a_1 t + a_2 t^2$, where a_0, a_1 and a_2 are the polynomial coefficients. Doing the mathematical calculations, we find

$$\int_{t_0}^{t_2} f(t)dt \approx \frac{T_s(n_1+n_2)}{6n_1 n_2}\left[(2n_1 n_2 - n_2{}^2)f(t_0) + (n_1 + n_2)^2 f(t_1) + (2n_1 n_2 - n_1{}^2)f(t_2)\right]. \quad (8)$$

For the case of n subintervals with a total number of $N_r = 2n + 1$ samples, and denoting by $n_{i1}$and n_{i2} the integers n_1 and n_2 for each subinterval respectively, we find that the total area of $f(t)$ from 0 to T can be estimated as:

$$\int_0^T f(t)dt \approx T_s \sum_{i=1}^{n} \frac{(n_{i1}+n_{i2})}{6n_{i1}n_{i2}}\left((2n_{i1}n_{i2} - n_{i2}{}^2)f(t_{2i-2}) + (n_{i1} + n_{i2})^2 f(t_{2i-1}) + (2n_{i1}n_{i2} - n_{i1}{}^2)f(t_{2i})\right). \quad (9)$$

B. Estimation Error

To find the estimation error for one subinterval, Er_{S3NS}, assuming that $T_s = h$ for the sake of simplicity, we start with:

$$Er_{\mathrm{S3NS}} = \int_0^{(n_1+n_2)h} f(t)dt - \frac{h(n_1+n_2)}{6n_1 n_2}\left[(2n_1 n_2 - n_2{}^2)f(t_0) + (n_1 + n_2)^2 f(t_1) + (2n_1 n_2 - n_1{}^2)f(t_2)\right]. \quad (10)$$

Now, the Taylor Series expansion of $f(t)$, $f(t_0)$, $f(t_1)$ and $f(t_2)$ at $t = t_1 = n_1 h$, is

$$f(t) = f(t_1) + (t - n_1 h)f^{(1)}(t_1) + \frac{1}{2}(t - n_1 h)^2 f^{(2)}(t_1) + \frac{1}{6}(t - n_1 h)^3 f^{(3)}(t_1) + \frac{1}{24}(t - n_1 h)^4 f^{(4)}(t_1) + O(t - n_1 h)^5. \quad (11)$$

$$f(t_0) = f(t_1) - n_1 h f^{(1)}(t_1) + \frac{1}{2}(n_1 h)^2 f^{(2)}(t_1) - \frac{1}{6}(n_1 h)^3 f^{(3)}(t_1) + \frac{1}{24}(n_1 h)^4 f^{(4)}(t_1) + O(n_1 h)^5. \quad (12)$$

$$f(t_1) = f(t_1). \quad (13)$$

$$f(t_2) = f(t_1) + n_2 h f^{(1)}(t_1) + \frac{1}{2}(n_2 h)^2 f^{(2)}(t_1) + \frac{1}{6}(n_2 h)^3 f^{(3)}(t_1) + \frac{1}{24}(n_2 h)^4 f^{(4)}(t_1) + O(n_2 h)^5. \quad (14)$$

Where $f^{(i)}(t_1)$ is the i-th derivative of $f(t)$ at $t = t_1$. Substituting (11)-(14) into (10), and carrying out some mathematical manipulation, we get

$$Er_{\mathrm{S3NS}} = \frac{h^4(n_1+n_2)^3(n_1-n_2)}{72} f^{(3)}(t_1) - \frac{h^5(n_1+n_2)^3(4n_1^2 - 7n_1 n_2 + 4n_2^2)}{720} f^{(4)}(t_1), \quad (15)$$

where a term of $O\left(F(n_1{}^6, n_2{}^6, h^6)\right)$ is neglected, since $F(.)$ is a function of fraction raised to the power of 6, which is very small compared to the other terms.

The error in (15) can be greatly decreased by choosing $n_1 = n_2$ (equally spaced samples), where it reduces to

$$Er_{\mathrm{S3NS},\, n_1 = n_2} = -\frac{n_1{}^5 h^5}{90} f^{(4)}(t_1). \quad (16)$$

This is exactly the same error of Simpson's 1/3 rule as found in literature for uniform sampling case, but without n_1, since $n_1 h$ here is the same as h in there, and both denote the spacing step between the uniform sampling points.

Note that there is a trade-off in selecting n_1 and n_2, where equal numbers means uniform sampling, and so, aliasing will occur if sampling frequency is less than Nyquist/Landau rate. Whereas choosing $n_1 \neq n_2$ means NUS, and this will mitigate aliasing effect, but also, will increase the error term accordingly.

Now, we calculate the total composite error for n subintervals, Er_{CS3NS}, which can be found by

$$Er_{\text{CS3NS}} = \sum_{i=1}^{n} \frac{h^4(n_{i1}+n_{i2})^3(n_{i1}-n_{i2})}{72} f^{(3)}(t_{2i-1}) - \frac{h^5(n_{i1}+n_{i2})^3(4n_{i1}^2-7n_{i1}n_{i2}+4n_{i2}^2)}{720} f^{(4)}(t_{2i-1}). \quad (17)$$

Finally, Table I shows an example for applying the CS3NS technique practically to obtaining one sample of the filtered output discrete-time signal.

TABLE I. APPLICATION OF CS3NS

Step 1	***Step 2***	***Step 3***	***Step 4***	***Step 5***
Signal's N_r nonuniform grid-based samples are acquired	Multiply sample amplitudes with N_r time-matching filter samples	Calculate AUC using (9)	Shift input samples one step	Repeat steps 2, 3 and 4 till the end of shifted samples

V. NUMERICAL RESULTS

We have designed a Kaiser-windowed bandpass FIR filter based on uniform grid sampling, as shown in Fig. 4 below. The uniform time spacing of the samples of the filter, T_s, is 0.25μs, i.e. F_s=4MHz. All other settings of the FIR filter are: Fstop1=65kHz, Fpass1=69kHz, Fpass2=81kHz, Fstop2=85 kHz, Bandwidth (BW)=15kHz and total number of samples (NF)= 3627. The input signal, $x(t)$, is a sum of five sinusoids with different frequencies: 13kHz, 69kHz, 75kHz, 81kHz and 90kHz. The FT of this signal, $X(f)$, is shown in Fig. 5a. So, the total bandwidth of this bandlimited signal is 90kHz. We want to filter this signal by suppressing the out-of-band components, keeping only those ones within the filter bandwidth, i.e. 69kHz, 75kHz and 81kHz. Therefore, the actual bandwidth of the filtered output signal should be 11kHz, in this case.

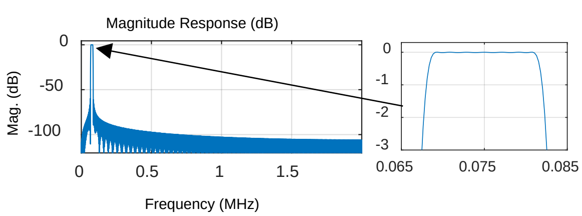

Fig. 4. The proposed uniformly oversampled bandpass FIR filter

Usually, an anti-aliasing analog filter is used before sampling the input signal to get rid of, or reduce to the least possible amount, any alias components that may foldback to the frequency range of consideration. Thanks to random sampling and filtering techniques, we can acquire the continuous-time input signal directly without using the analog filter, and attenuating the undesired alias frequency components dramatically. Actually, this is the powerful point of using nonuniform and random sampling approach, and hence the name, digital alias-free signal processing.

Grid-based NUS of the input signal is, then, carried out using ToRa, StSa and AnSt schemes within an observation time window T=12ms. In our simulations, we have also included the results of the CS3NS interpolation technique based on sampling points selected same as StSa's ones, since interpolating such points has shown better results than ToRa and AnSt for most of our simulations (not included here). Indeed, AnSt yields better results than StSa if the sampled signal is smooth, i.e. monotonically increasing or decreasing within each stratum (or sub-integral area), which is not usually the case when using sub-Nyquist NUS approach due to the scarcity of points across the observed time window. Otherwise, it has no big advantage over StSa.

The FIR filter is used to filter the randomly sampled signal. For each single sampling technique and each specific number of NUS points, we have carried out 10 independent simulations to obtain averaged and smoothed results. Then, the FTs of the estimated filtered output signals are compared to the FT of the reference signal (densely and uniformly oversampled copy of the input signal filtered by the same FIR filter), $Y_u(f)$, as shown in Fig. 5b. The FT spectra of the nonuniform filtered output signals are estimated by applying the same approach of random sampling technique used initially to obtain these filtered signals. This means adding a second layer of randomized signal processing: filtering and FT spectrum estimating.

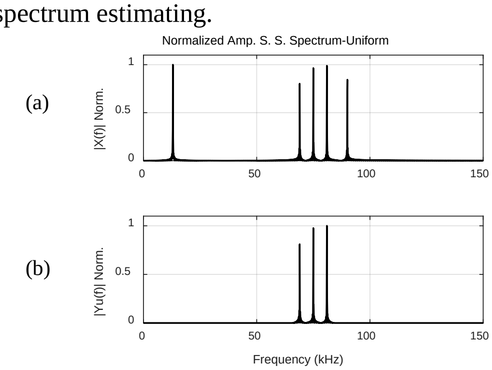

Fig. 5. Normalized single-sided FT spectra of uniform (a) input signal, $X(f)$, and (b) filtered output signal, $Y_u(f)$, which used as a reference.

To compare the estimated FT spectrum of a given output signal, for example $Y_n(f)$, with that of the reference signal, $Y_u(f)$ obtained using MATLAB's built-in CONV and FFT functions, we calculate the NRMSE using this formula:

$$\text{NRMSE} = \frac{\sqrt{\frac{\sum_{i=1}^{L}|Y_n(f_i)-Y_u(f_i)|^2}{L}}}{\max(Y_u)-\min(Y_u)}, \quad (18)$$

where L is the number of FT frequency components within the range [Fstop1, Fstop2]. This range was selected since the FT of $Y_u(f)$ is a priori known to be zero or so outside these filter stop bands, therefore, any statistical errors or small aliases that might appear outside this range are negligible. Moreover, we choose the normalization approach to guarantee scaling consistency over different sampling and interpolation schemes and simulation settings.

The simulation results are depicted in Fig. 6, starting with only 200 NUS points ($F_{s,av,NUS}$= 200/12ms≈16.67kHz and ending with 8000 points ($F_{s,av,NUS}$= 8000/12ms≈0.67MHz). We notice that, initially, StSa and AnSt yield less estimation errors than both ToRa and CS3NS. However, by increasing the number of points gradually we can clearly see that the convergence rate of CS3NS interpolation technique is much faster than all other sampling techniques. In addition, at a specific number of points (>3200), CS3NS's absolute NRMSE starts to get smaller than those for StSa and AnSt, as seen in Fig. 7. That is because the interpolation errors decrease dramatically with the increase of used sampling points, but of course, on the price of increasing the computational cost.

It is worth noting that the original input signal bandwidth is 90kHz. So, if traditional uniform sampling techniques are to be used instead of nonuniform ones, then either a) we have to use an analog anti-aliasing filter, and then uniformly sampling the analog output by at least 24kHz, which is the Landau rate required for the 12kHz bandpass signal [69kHz, 81kHz], or b) directly sampling the 90kHz-bandwidth signal with 180kHz Nyquist rate to avoid aliasing issues. However, it is possible to use NUS techniques, StSa for example, and get feasible results without even using the analog anti-aliasing filter beforehand. Fig. 8 shows the normalized single-sided estimated FT for the filtered output signal based on average NUS rate $F_{s,av,NUS}$= 20kHz, where the sinusoids within the bandpass frequency range are easily distinguishable.

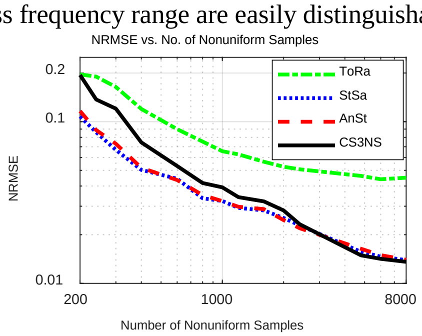

Fig. 6. NRMSE of the estimated FT spectra of the filtered outputs vs. number of used nonuniform samples.

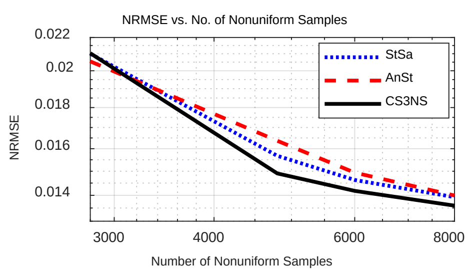

Fig. 7. CS3NS's NRMSE gets smaller than others after 3.2k samples.

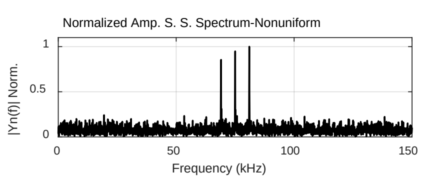

Fig. 8. Estimated FT of the filtered output signal using 20kHz average sampling frequency StSa-based nonuniform sampling scheme.

VI. Conclusion

It has been shown that filtering irregularly sampled signals is possible even by using average sampling frequency less than the required Nyquist rate. The random samples are acquired based on several random sampling techniques, such as: ToRa, StSa and AnSt. Furthermore, a new interpolation-based technique, CS3NS, has been proposed, where it is used as a numerical integration technique to calculate the area under the curve for both filter convolution and FT estimation. Apart from minor statistical errors, alias frequency components have been eliminated or reduced dramatically using an appropriate number of nonuniform samples and a given random sampling technique. This is illustrated by being able to detect output signal FT with $F_{s,av,NUS} \ll f_{Nyq}$, which is can't be accomplished by using uniform sampling.

Finally, simulation results show that uniform convergence rate of the new CS3NS interpolation technique is faster than any other discussed ones, despite the fact that the absolute NRMSE values of CS3NS is larger for a small number of sampling points. But the uniform convergence rate is more crucial metric than the absolute error values in such cases.

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