Comparison Between Uniform and Nonuniform Interpolation Techniques for Digital Alias-free FIR Filtering

Hikmat Y. Darawsheh, Andrzej Tarczynski
University of Westminster
London, UK
hikmat.darawsheh@my.westminster.ac.uk, tarczya@westminster.ac.uk

ABSTRACT

In this paper, we propose three grid-based nonuniform interpolation techniques to find the AUC of the convolution operation of a digital alias-free FIR filter. Up to the authors' knowledge, these techniques were not addressed in literature before. We call them composite 3-nonuniform-sample (C3NS), composite 4-nonuniform-sample (C4NS) and composite 5-nonuniform-sample (C5NS) rules. They are named after the traditional composite Simpson's 1/3 rule which is usually used in second-order polynomial interpolation of equally-spaced sampling points. The proposed new rules show better estimated results than the uniform-based ones when the number of sampling points doesn't match the required Nyquist rate. Moreover, we prove that composite Simpson's 1/3 rule is more accurate than composite Simpson's 3/8 rule mathematically and by simulation.

Keywords

Nonuniform interpolation, random sampling, digital alias-free signal processing, FIR filter, composite Simpson's rules, Boole's rule

1. INTRODUCTION

Sampling theory in signal processing was historically addressed by Cauchy's work back to 1841 [1], although other similar works, like interpolation theory, have been carried out by many big mathematician names such as Gauss, Newton, Lagrange and others [2]. In order to sample an analog signal properly, the sampling frequency should be selected carefully to enable the perfect reconstruction of the original signal. In uniform sampling, the samples of the signal are taken evenly at integer multiples of a constant time period, T_s, known as the sampling time.

The Shannon sampling theorem [3]-[4], states that a bandlimited baseband signal with a maximum frequency component, f_m, must be sampled at a frequency not less than twice f_m, i.e. $F_s \geq 2f_m$, for the reconstruction process to be carried out unambiguously. This minimum sampling frequency is also known as the Nyquist rate. Although this is a sufficient but not necessary condition, aliasing occurs if the sampling frequency is less than the Nyquist rate using uniform sampling in classical digital signal processing (DSP), where spectrum replicas of the signal fold back to the band of consideration, and the resulting spectrum is no more reflecting the actual Fourier transform of the original signal, and so, it can't be reconstructed perfectly.

The notion of nonuniform sampling of continuous-time signals to avoid aliasing has been indirectly addressed by many researchers on the early decades of last century [5]. But it was until 1956 that a dedicated engineering-based paper on nonuniform sampling (interpolation) has been published [6]. In addition to his other paper [7], published in 1957, Yen has derived mathematical expressions for interpolating specific irregular and nonuniform samples. Later on, in 1960, Shapiro and Silverman [8] showed that alias-free sampling could be performed with sampling frequency less than the Nyquist rate.

Digital alias-free signal processing (DASP) [9] is a fairly novel notion of processing signals digitally without the effect of aliasing, even if the class of processed signals is not heavily restricted. This apparently impossible objective is achieved by using suitable random (or nonuniform) sampling schemes.

For a given random sampling scheme and a deterministic continuous-time signal, the resulting discrete-time signal becomes a random signal. It has been shown that it is possible to select sampling schemes such that any two different continuous-time signals are rasped at two different random discrete-time signals [10]. In that sense the sampling schemes and subsequent signal processing are alias-free.

Filtering of nonuniformly sampled input signals has been addressed in [11], and in [12] an underlying uniform grid is used to align filter time response samples to the input signal's ones. This paper is an extension to [12] including the mathematical derivation of two extra interpolation rules, called C4NS and C5NS, in addition to comparison between some interpolation techniques that use uniform sampling methods. Moreover, the full error analysis for the composite 3-nonuniform-sample (C3NS) rule is also derived here.

The rest of this paper is composed of three sections addressing filtering and interpolation for uniform and nonuniform sampling, along with computer simulation and numerical results. Conclusion is also provided at the end of the paper.

2. UNIFORM SAMPLING APPROACH

2.1 Digital Filtering

In classical DSP, the discrete convolution operation of digital filtering, using uniform sampling frequency, F_s, and N sampling points within T seconds observation period, is

$$y(t) = \frac{T}{N} \sum_{n=0}^{N-1} x(nT_s)\, h(t - nT_s). \tag{1}$$

where $x(nT_s)$ represents the sampled input signal, $h(nT_s)$ is the filter time response, and $T_s = 1/F_s$ is the sampling time. Now, by considering values of $y(t)$ at discrete time instants that are k multiples of the sampling time, and taking into account that $T = NT_s$, we get

$$y(kT_s) = T_s \sum_{n=0}^{N-1} x(nT_s)\, h(kT_s - nT_s). \tag{2}$$

Equation (2) is nothing more than finding an AUC of a specific product using the simple rectangle rule with T_s representing the incremental time value of the calculation process. This rule, sometimes referred to as the midpoint rule, is simple and easy to implement. However, it is not the perfect choice of calculating the convolution operation always.

2.2 Interpolation

Interpolation is a mathematical technique used in curve fitting of a set of data points or discrete samples. In linear interpolation, for example, a first order polynomial is used to calculate function values at intermediate points between two given data points. While estimating the function values outside these points is known as extrapolation. In this section, we focus on interpolation of equally-spaced points using higher order Lagrange polynomials.

2.2.1 Composite Simpson's 1/3 Rule

In regular Simpson's 1/3 rule, a given function, $f(t)$, can be approximated by using a second-order Lagrange polynomial, $P(t)$. This polynomial is formed by means of quadratic interpolation techniques and making use of only three equally spaced samples $\{(t_0, f(t_0)), (t_1, f(t_1)), (t_2, f(t_2))\}$, i.e. the boundaries of two equal segments of $f(t)$, as shown in Fig.1. Once the parabolic equation becomes known, we can estimate the area under $f(t)$ within an interval $[a, b]$ by integrating $P(t)$ from a to b. It has been found that

$$\int_a^b f(t)dt \approx \frac{h}{3}(f_0 + 4f_1 + f_2), \quad (3)$$

where, for simplicity, $f_0 = f(t_0)$, $f_1 = f(t_1)$ and $f_2 = f(t_2)$. Note that $t_0 = a$, $t_1 = \frac{b+a}{2}$, $t_2 = b$, and $h = \frac{b-a}{2}$ is the spacing step between any two consecutive samples, or equivalently, the segment's width.

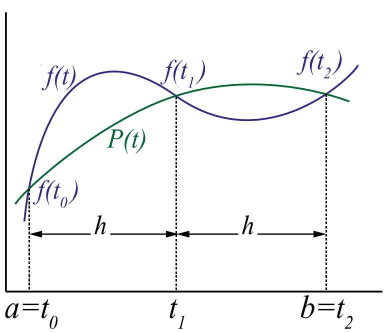

Fig. 1. Simpson's 1/3 rule, where $P(t)$ is a fitting curve for $f(t)$ using only three equally-spaced samples of $f(t)$.

The error of estimation in Simpson's 1/3 rule is bounded by the maximum value of the expression,

$$Err_{Simp1/3} = -\frac{h^5}{90} f^{(4)}(\xi), \quad (4)$$

where $f^{(4)}$ is the forth derivative of $f(t)$ and ξ is a number within the open interval (a, b).

If the finite interval $[a, b]$ is relatively large, then the estimation error will be high. Thanks to the composite (or extended) Simpson's 1/3 rule, where the interval $[a, b]$ is divided into n equal subintervals, with two identical segments per each subinterval, within which we can apply the regular Simpson's 1/3 rule, explained above, in each single subinterval as shown in Fig. 2. In this case we have $N = 2n + 1$ equally spaced samples with step segment width $h = \frac{b-a}{N-1} = \frac{b-a}{2n}$. Integration of $f(t)$ using composite Simpson's 1/3 rule can be numerically estimated as,

$$\int_a^b f(t)dt \approx \frac{h}{3}\left(f_0 + 4\sum_{i=1}^{n} f_{2i-1} + 2\sum_{i=1}^{n-1} f_{2i} + f_{N-1}\right), \quad (5)$$

and the error of estimation is,

$$Err_{CSimp1/3} = -\frac{h^5}{90}\sum_{i=1}^{n} f^{(4)}(\xi_i), \quad (6)$$

where $\xi_i \in (t_{2i-2}, t_{2i})$. We may rewrite (6) in terms of the average estimated error across all subintervals, $\overline{f^{(4)}(\xi)}$, as

$$Av.Err_{CSimp1/3} = -\frac{h^4(b-a)}{180}\overline{f^{(4)}(\xi)} = -\frac{(b-a)^5}{180(N-1)^4}\overline{f^{(4)}(\xi)}, \quad (7)$$

where $\xi \in [a, b]$.

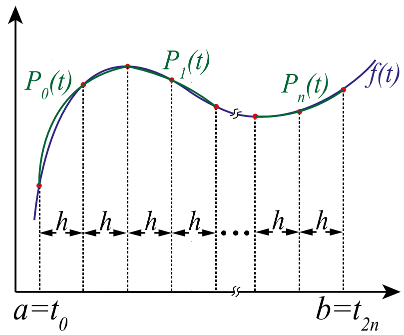

Fig. 2. Composite Simpson's 1/3 rule with n parabolic interpolation functions employed.

2.2.2 Composite Simpson's 3/8 Rule

In much similar way, uniform Simpson's 3/8 rule uses four equidistant points interpolation, with spacing segment width $h = \frac{b-a}{3}$, to estimate the function $f(t)$ within the interval $[a, b]$, based on third-order Lagrange polynomial. It has been found that the area under $f(t)$ from a to b can be estimated as

$$\int_a^b f(t)dt \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3), \quad (8)$$

with an estimation error of $Err_{Simp3/8} = -\frac{3h^5}{80} f^{(4)}(\xi)$, where $f^{(4)}$ is the forth derivative of $f(t)$ and ξ is a number within the open interval (a, b).

For more accurate estimation of fining the total area under the curve of $f(t)$, the composite Simpson's 3/8 rule suggests partitioning the interval $[a, b]$ into n subintervals with $3n$ segments per each subinterval, then, interpolating every four samples within each subinterval by applying the Simpson's 3/8 rule, as illustrated above, and thence adding the integral results of all subintervals together. Therefore, a total of $N = 3n + 1$ equally spaced samples $(t_0 = a, t_1, t_2, \ldots, t_i = a + ih, \ldots, t_{N-1} = b)$, with segment width $h = \frac{b-a}{N-1} = \frac{b-a}{3n}$, are considered. The new approximation of $f(t)$'s definite integral is

$$\int_a^b f(t)dt \approx \frac{3h}{8}\left(f_0 + 3\sum_{i=1}^{n}(f_{3i-2} + f_{3i-1}) + 2\sum_{i=1}^{n-1} f_{3i} + f_{N-1}\right), \quad (9)$$

and the error of estimation is $Err_{CSimp3/8} = -\frac{3h^5}{80}\sum_{i=1}^{n} f^{(4)}(\xi_i)$, where $\xi_i \in (t_{3i-3}, t_{3i})$. Analogously to composite Simpson's 1/3 rule, the average error can be estimated as

$$Av.Err_{CSimp3/8} = -\frac{h^4(b-a)}{80}\overline{f^{(4)}(\xi)} = -\frac{(b-a)^5}{80(N-1)^4}\overline{f^{(4)}(\xi)}. \quad (10)$$

By comparing the average errors in (7) and (10), we can deduce that the composite Simpson's 1/3 rule is better in estimating the area under the curve of a given function, $f(t)$, than the composite Simpson's 3/8 rule within the same interval $[a, b]$ and the same number of equispaced samples, assuming that the average fourth derivatives of $f(t)$ for both rules, evaluated at a specific number between a and b, are almost the same. This motivates us to explore the composite Boole's rule and check its associated error term.

Now, an emerging question is: if the interpolation using composite Simpson's 1/3 rule seems to be more accurate than the composite Simpson's 3/8 rule, so why do we need the last one? The simple answer is that, sometimes, we don't have the proper number of samples to apply the first one alone. To overcome this limitation, we use a mix of the two rules together. For example, if there is an even number of samples, or the number of samples is odd but does not satisfy the condition of having integer value for the number of subintervals in the formula: $N = 3n + 1$, then there is a necessity to mix more than one interpolation rule.

2.2.3 Composite Boole's 3/8 Rule

Five uniformly spaced function evaluations (samples) are considered for the case of Boole's rule, with segment spacing $h = \frac{b-a}{4}$. Therefore, interpolating $f(t)$ to calculate the area under the curve from a to b yields

$$\int_a^b f(t)dt \approx \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4), \quad (11)$$

where $t_i = a + ih, i = 0,1,2,3,4$. The error associated with this approximation is $Err_{\text{Boole}} = -\frac{8h^7}{945} f^{(6)}(\xi),\ \ \xi \in (a, b)$.

With regards to the composite Boole's rule where the interval $[a, b]$ is divided into n equal subintervals with $4n$ segments per subinterval, a total of $N = 4n + 1$ uniform samples ($t_0 = a, t_1, t_2, \ldots, t_i = a + ih, \ldots, t_{N-1} = b$), with segment's width $h = \frac{b-a}{N-1} = \frac{b-a}{4n}$, are considered. Therefore, the definite integral of $f(t)$ can be estimated as:

$$\int_a^b f(t)dt \approx \frac{2h}{45}\sum_{i=1}^{n}(7f_{4i-4} + 32f_{4i-3} + 12f_{4i-2} + 32f_{4i-1} + 7f_{4i}). \quad (12)$$

The error of estimation is $Error_{CBoole} = -\frac{8h^7}{945}\sum_{i=1}^{n} f^{(6)}(\xi_i)$, where $\xi_i \in (t_{4i-4}, t_{4i})$. While the overall average error is estimated by

$$Av.Err_{CBoole} = -\frac{2h^6(b-a)}{945}\overline{f^{(6)}(\xi)} = -\frac{2(b-a)^7}{945(N-1)^6}\overline{f^{(6)}(\xi)}. \quad (13)$$

It is not easy to compare the absolute values of the average errors for composite Boole's rule and composite Simpson's 1/3 rule (or composite Simpson's 3/8 rule) mathematically. However, we know, at least, that the rate of uniform convergence of the average error term in composite Boole's rule, (13), is faster than the other ones, (7) and (10), for fixed interval limits, since it is proportional to $(N-1)^{-6}$ compared to $(N-1)^{-4}$ for the other ones.

3. NONUNIFORM SAMPLING APPROACH

Lagrange interpolation polynomials can be generalized to include unequally spaced nodes (samples). In general, any continuous and differentiable function, $f(t)$, with N nonuniformly distributed nodes $\{(t_0, f_0), (t_1, f_1), \ldots, (t_{N-1}, f_{N-1})\}$, can be approximated with a general $(N-1)$-degree piecewise polynomial $P_{N-1}(t) = \sum_{k=0}^{N-1} P_k(t)$, where each $P_k(t)$ is calculated by

$$P_k(t) = f_k \prod_{\substack{i=0 \\ i\neq k}}^{N-1} \frac{t-t_i}{t_k-t_i}. \quad (14)$$

3.1 Composite 3-Nonuniform-Sample Rule

The proposed composite 3-nonuniform-sample (C3NS) rule addresses the case of interpolation of nonuniform sampling points that are aligned to an underlying uniform grid. This was chosen in purpose, where a potential practical implementation is in mind, and to mitigate the aliasing problem by using random-based function evaluations or nodes.

3.1.1 Estimated Area Under the Curve (AUC)

Suppose we have a continuous-time function, $f(t)$, and we need to estimate the area under $f(t)$ within the interval $[0, T)$ by interpolating specific number of nonuniformly spaced samples of it. Hence, we divide the whole interval $[0, T)$ into subintervals according to the number of sampling points. Every three consecutive samples constitute one subinterval, as shown in Fig. 3. Left and right samples (borders) of a given subinterval are shared with previous and next subintervals to form the composite rule and to calculate the total interpolated area from 0 to T.

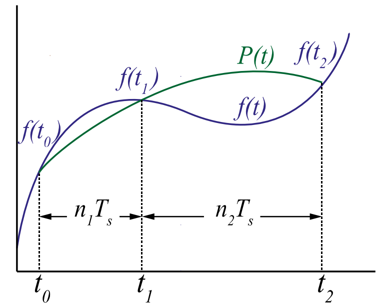

Fig. 3. One subinterval, $[t_0, t_2]$, of the C3NS rule.

Starting with the first three nonuniform samples, at time instants $t_0 = 0$, t_1 and t_2 (one subinterval), and recalling that time spacing between any two samples is integer multiple of T_s, the uniform grid time step, we introduce the integer numbers n_1 and n_2 as follows: $t_1 - t_0 = n_1 T_s$ and $t_2 - t_1 = n_2 T_s$. Now, we estimate the subinterval area under $f(t)$ from t_0 to $t_2 = (n_1 + n_2)T_s$ by simply interpolating the three sample points $f_0 = f(t_0)$, $f_1 = f(t_1)$ and $f_2 = f(t_2)$ using a second-order Lagrange polynomial $P(t)$. Working out the mathematical calculations, we find the area within this subinterval to be equal to

$$\int_{t_0}^{t_2} f(t)dt \approx \frac{T_s(n_1 + n_2)}{6n_1n_2}[(2n_1n_2 - n_2^2)f_0 + (n_1 + n_2)^2 f_1 + (2n_1n_2 - n_1^2)f_2]. \quad (15)$$

For the case of n subintervals with a total number of $N = 2n + 1$ samples, and denoting by n_{i1} and n_{i2} the integers n_1 and n_2 for each subinterval respectively, then the total area of $f(t)$ from 0 to T can be estimated as:

$$\int_0^T f(t)dt \approx T_s \sum_{i=1}^{n} \frac{(n_{i1} + n_{i2})}{6n_{i1}n_{i2}}\left((2n_{i1}n_{i2} - n_{i2}^2)f_{2i-2} + (n_{i1} + n_{i2})^2 f_{2i-1} + (2n_{i1}n_{i2} - n_{i1}^2)f_{2i}\right). \quad (16)$$

3.1.2 Error Analysis

To find the estimation error for one subinterval, $Err_{3\text{NS}}$, assuming that $T_s = h$ for the sake of simplicity (and to be inline with literature's notation as much as possible), we start with:

$$Err_{3\text{NS}} = \int_0^{(n_1+n_2)h} f(t)dt - \frac{h(n_1 + n_2)}{6n_1n_2}[(2n_1n_2 - n_2^2)f_0 + (n_1 + n_2)^2 f_1 + (2n_1n_2 - n_1^2)f_2]. \quad (17)$$

Now, the Taylor Series expansion of $f(t)$, f_0, f_1 and f_2 at $t = t_1 = n_1 h$, is

$$f(t) = f_1 + (t - n_1h)f_1^{(1)} + \frac{1}{2}(t - n_1h)^2 f_1^{(2)} + \frac{1}{6}(t - n_1h)^3 f_1^{(3)} + \frac{1}{24}(t - n_1h)^4 f_1^{(4)} + O(t - n_1h)^5, \quad (18)$$

where $f_1^{(i)}$ is the i-th derivative of $f(t)$ at $t = t_1$.

$$f_0 = f(t_1) - n_1 h f^{(1)}(t_1) + \frac{1}{2}(n_1 h)^2 f^{(2)}(t_1) - \frac{1}{6}(n_1 h)^3 f^{(3)}(t_1) + \frac{1}{24}(n_1 h)^4 f^{(4)}(t_1) + O(n_1 h)^5. \quad (19)$$

$$f_1 = f_1. \quad (20)$$

$$f_2 = f_1(t_1) + n_2 h f_1^{(1)} + \frac{1}{2}(n_2 h)^2 f_1^{(2)} + \frac{1}{6}(n_2 h)^3 f_1^{(3)} + \frac{1}{24}(n_2 h)^4 f_1^{(4)} + O(n_2 h)^5. \quad (21)$$

Substituting (18)-(21) into the expression of $Err_{3\text{NS}}$ shown in (17), we get

$$Err_{3\text{NS}} = \int_0^{(n_1+n_2)h} \Big[f_1 + (t - n_1 h) f_1^{(1)} + \frac{1}{2}(t - n_1 h)^2 f_1^{(2)} + \frac{1}{6}(t - n_1 h)^3 f_1^{(3)} + \frac{1}{24}(t - n_1 h)^4 f_1^{(4)} + O(t - n_1 h)^5\Big] dt - \frac{h(n_1+n_2)}{6 n_1 n_2}\Big[(2n_1 n_2 - n_2^2)\Big(f(t_1) - n_1 h f^{(1)}(t_1) + \frac{1}{2}(n_1 h)^2 f^{(2)}(t_1) - \frac{1}{6}(n_1 h)^3 f^{(3)}(t_1) + \frac{1}{24}(n_1 h)^4 f^{(4)}(t_1) + O(n_1 h)^5\Big) + (n_1 + n_2)^2 f_1 + (2n_1 n_2 - n_1^2)\Big(f_1(t_1) + n_2 h f_1^{(1)} + \frac{1}{2}(n_2 h)^2 f_1^{(2)} + \frac{1}{6}(n_2 h)^3 f_1^{(3)} + \frac{1}{24}(n_2 h)^4 f_1^{(4)} + O(n_2 h)^5\Big)\Big]. \quad (22)$$

$$Err_{3\text{NS}} = (n_1 + n_2) h f_1 + \frac{1}{2}(n_2^2 - n_1^2) h^2 f_1^{(1)} + \frac{1}{6}(n_2^3 + n_1^3) h^3 f_1^{(2)} + \frac{1}{24}(n_2^4 - n_1^4) h^4 f_1^{(3)} + \frac{1}{120}(n_2^5 + n_1^5) h^5 f_1^{(4)} + O\big((n_2^6 - n_1^6) h^6\big) - \frac{h(n_1+n_2)}{6 n_1 n_2}\Big[(2n_1 n_2 - n_2^2)\Big(f_1 - n_1 h f_1^{(1)} + \frac{1}{2}(n_1 h)^2 f_1^{(2)} - \frac{1}{6}(n_1 h)^3 f_1^{(3)} + \frac{1}{24}(n_1 h)^4 f_1^{(4)} + O(n_1 h)^5\Big) + (n_1 + n_2)^2 f_1 + (2n_1 n_2 - n_1^2)\Big(f_1 + n_2 h f_1^{(1)} + \frac{1}{2}(n_2 h)^2 f_1^{(2)} + \frac{1}{6}(n_2 h)^3 f_1^{(3)} + \frac{1}{24}(n_2 h)^4 f_1^{(4)} + O(n_2 h)^5\Big)\Big]. \quad (23)$$

Arranging the terms and carrying out some mathematical manipulation, we get

$$Err_{3\text{NS}} = \frac{h^4 (n_1 + n_2)^3 (n_1 - n_2)}{72} f_1^{(3)} - \frac{h^5 (n_1 + n_2)^3 (4n_1^2 - 7n_1 n_2 + 4n_2^2)}{720} f_1^{(4)}, \quad (24)$$

where a term of $O\big(F(n_1{}^6, n_2{}^6, h^6)\big)$ is neglected, since $F(.)$ is a function of fraction raised to the power of 6, which is very small compared to the other terms.

The error in (24) can be greatly decreased by choosing $n_1 = n_2$ (equally spaced samples), where it reduces to

$$Err_{3\text{NS},\, n_1 = n_2} = -\frac{n_1{}^5 h^5}{90} f_1^{(4)}. \quad (25)$$

This is exactly the same error for Simpson's 1/3 rule as found in literature for uniform sampling case, but without n_1, since $n_1 h$ here is the same as h in there, and both denote the spacing step between the uniform sampling points.

Note that there is a trade-off in selecting n_1 and n_2, where equal numbers mean uniform sampling, and so, aliasing will occur when sampling frequency is less than Nyquist/Landau rate. Whereas choosing $n_1 \neq n_2$ means NUS, and this will mitigate aliasing effect, but also, will increase the error term accordingly.

Now, we calculate the total composite error for n subintervals, Err_{C3NS}, which can be found by

$$Err_{\text{C3NS}} = \sum_{i=1}^{n} \frac{h^4 (n_{i1} + n_{i2})^3 (n_{i1} - n_{i2})}{72} f_{2i-1}{}^{(3)} - \frac{h^5 (n_{i1} + n_{i2})^3 (4n_{i1}^2 - 7n_{i1} n_{i2} + 4n_{i2}^2)}{720} f_{2i-1}{}^{(4)}. \quad (26)$$

3.2 Composite 4-Nonuniform-Sample Rule

This proposed composite 4-nonuniform-sample (C4NS) rule interpolates four nonuniform sampling points, as follows

$$\int_a^b f(t)dt \approx \sum_{i=0}^{3} w_i f_i, \quad (27)$$

where w_i is the weight of the i-th function point (sample), and $n_i h$ is the distance between the samples f_{i-1} and f_i,

$$w_0 = \frac{h\,(3n_1^3 + 5n_1^2 n_2 + n_1^2 n_3 + n_1 n_2^2 - n_1 n_3^2 - n_2^3 - n_2^2 n_3 + n_2 n_3^2 + n_3^3)}{12 n_1 (n_1 + n_2)}. \quad (28)$$

$$w_1 = \frac{h\,(n_1 + n_2 - n_3)(n_1 + n_2 + n_3)^3}{12 n_1 n_2 (n_2 + n_3)}. \quad (29)$$

$$w_2 = \frac{h\,(n_2 - n_1 + n_3)(n_1 + n_2 + n_3)^3}{12 n_2 n_3 (n_1 + n_2)}. \quad (30)$$

$$w_3 = \frac{h\,(n_1^3 + n_1^2 n_2 - n_1^2 n_3 - n_1 n_2^2 + n_1 n_3^2 - n_2^3 + n_2^2 n_3 + 5n_2 n_3^2 + 3n_3^3)}{12 n_3 (n_2 + n_3)}. \quad (31)$$

3.3 Composite 5-Nonuniform-Sample Rule

In composite 5-nonuniform-sample (C5NS) rule, we use five nonuniform sampling points to estimate $f(t)$, so

$$\int_a^b f(t)dt \approx \sum_{i=0}^{4} w_i f_i, \quad (32)$$

where

$$w_0 = -\frac{h}{60\, n_1 (n_1 + n_2)(n_1 + n_2 + n_3)}(-12\, n_1^4 - 33\, n_1^3 n_2 - 18\, n_1^3 n_3 - 3\, n_1^3 n_4 - 27\, n_1^2 n_2^2 - 29\, n_1^2 n_2 n_3 - 4\, n_1^2 n_2 n_4 - 2\, n_1^2 n_3^2 + n_1^2 n_3 n_4 + 3\, n_1^2 n_4^2 - 3\, n_1 n_2^3 - 4\, n_1 n_2^2 n_3 + n_1 n_2^2 n_4 + n_1 n_2 n_3^2 + 2\, n_1 n_2 n_3 n_4 + n_1 n_2 n_4^2 + 2\, n_1 n_3^3 + n_1 n_3^2 n_4 - 4\, n_1 n_3 n_4^2 - 3\, n_1 n_4^3 + 3\, n_2^4 + 7\, n_2^3 n_3 + 2\, n_2^3 n_4 + 3\, n_2^2 n_3^2 + n_2^2 n_3 n_4 - 2\, n_2^2 n_4^2 - 3\, n_2 n_3^3 - 4\, n_2 n_3^2 n_4 + n_2 n_3 n_4^2 + 2\, n_2 n_4^3 - 2\, n_3^4 - 3\, n_3^3 n_4 + 3\, n_3^2 n_4^2 + 7\, n_3 n_4^3 + 3\, n_4^4). \quad (33)$$

$$w_1 = \frac{h(n_1 + n_2 + n_3 + n_4)^3}{60\, n_1 n_2 (n_2 + n_3)(n_2 + n_3 + n_4)}(3\, n_1^2 + 6\, n_1 n_2 + n_1 n_3 - 4\, n_1 n_4 + 3\, n_2^2 + n_2 n_3 - 4\, n_2 n_4 - 2\, n_3^2 + n_3 n_4 + 3\, n_4^2). \quad (34)$$

$$w_2 = -\frac{h(n_1 + n_2 + n_3 + n_4)^3}{60\, n_2 n_3 (n_1 + n_2)(n_3 + n_4)}(3\, n_1^2 + n_1 n_2 + n_1 n_3 - 4\, n_1 n_4 - 2\, n_2^2 - 4\, n_2 n_3 + n_2 n_4 - 2\, n_3^2 + n_3 n_4 + 3\, n_4^2). \quad (35)$$

$$w_3 = \frac{h(n_1 + n_2 + n_3 + n_4)^3}{60\, n_3 n_4 (n_2 + n_3)(n_1 + n_2 + n_3)}(3\, n_1^2 + n_1 n_2 - 4\, n_1 n_3 - 4\, n_1 n_4 - 2\, n_2^2 + n_2 n_3 + n_2 n_4 + 3\, n_3^2 + 6\, n_3 n_4 + 3\, n_4^2). \quad (36)$$

$$w_4 = -\frac{h}{60\, n_4 (n_3 + n_4)(n_2 + n_3 + n_4)}(3\, n_1^4 + 7\, n_1^3 n_2 + 2\, n_1^3 n_3 - 3\, n_1^3 n_4 + 3\, n_1^2 n_2^2 + n_1^2 n_2 n_3 - 4\, n_1^2 n_2 n_4 - 2\, n_1^2 n_3^2 + n_1^2 n_3 n_4 + 3\, n_1^2 n_4^2 - 3\, n_1 n_2^3 - 4\, n_1 n_2^2 n_3 + n_1 n_2^2 n_4 + n_1 n_2 n_3^2 + 2\, n_1 n_2 n_3 n_4 + n_1 n_2 n_4^2 + 2\, n_1 n_3^3 + n_1 n_3^2 n_4 - 4\, n_1 n_3 n_4^2 - 3\, n_1 n_4^3 - 2\, n_2^4 - 3\, n_2^3 n_3 + 2\, n_2^3 n_4 + 3\, n_2^2 n_3^2 + n_2^2 n_3 n_4 - 2\, n_2^2 n_4^2 + 7\, n_2 n_3^3 - 4\, n_2 n_3^2 n_4 - 29\, n_2 n_3 n_4^2 - 18\, n_2 n_4^3 + 3\, n_3^4 - 3\, n_3^3 n_4 - 27\, n_3^2 n_4^2 - 33\, n_3 n_4^3 - 12\, n_4^4). \quad (37)$$

It is very complicated to work out the error terms in C4NS and C5NS rules mathematically, hence, it will be skipped for the time being!

4. Simulation Results

We have carried out two sets of numerical integration using uniform sampling- and nonuniform sampling-based interpolation methods discussed above. Fig.4 shows the first set of results for calculating AUC of a test signal $f(t) = sin(t) + cos(2t)$ within the interval [0,1.2] seconds. It is obvious that uniform-based interpolation techniques, depicted in solid lines, outperform their counterparts, i.e. same-order Lagrange polynomials ones, shown in dashed lines, for the whole range of sampling points from 61 to 601 samples. This is because of the low frequency used in the signal (less than 1Hz.) Note also that CS13 results are more accurate than CS38, and this emphasize our conclusion on the average errors in (7) and (10).

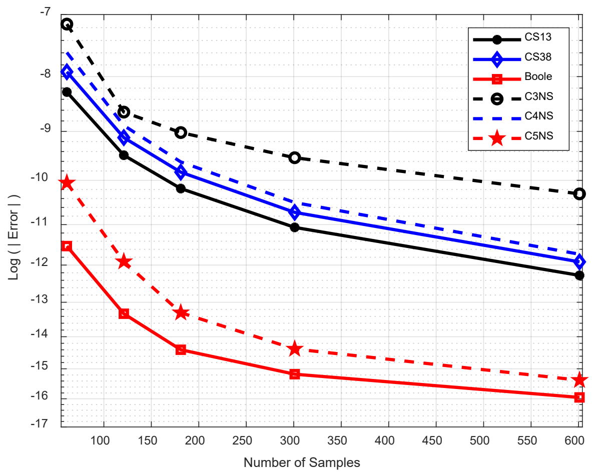

Fig. 4. Error analysis for input signal of only two single sinusoids having frequencies $1/(2\pi)$ Hz and $1/\pi$ Hz.

In the second set of simulations, we have used a much complex signal that has higher frequency components. Namely, $f(t) = 10^{-2}\, sinc\left(50\left(t - \frac{T}{2}\right)\right) \cos(2\pi 10^3 t) + 10^{-2}\, sinc\left(273\left(t - \frac{T}{2}\right)\right) cos(6\pi 10^3 t)$. As depicted in Fig. 5.

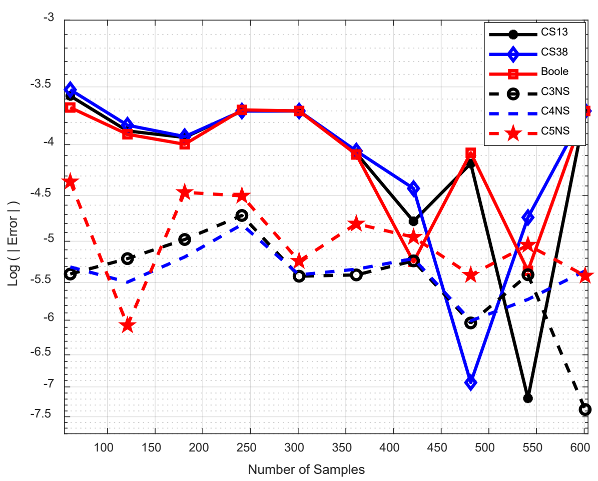

Fig. 5. Error analysis for input signal of two sinc functions shifted at two different carrier frequencies.

Fig 5 shows that the nonuniform interpolation techniques now outperforms the uniform ones in the left part of the figure, where the number of samples is equivalent to an average sampling frequency less than the required sampling rate.

5. Conclusion

We have derived three mathematical formulas for interpolation of grid-based nonuniform samples (C3NS, C4NS and C5NS), as well as the full error analysis of the C3NS rule. Moreover, a mathematical comparison between uniform sampling-based interpolation techniques is presented, where it has been shown that for the same number of sampling points and same interpolation interval, composite Boole's rule uniform convergence rate is much faster than other uniform composite rules' ones, and both CS13 and CS38 have the same convergence rate, with CS13 outperforms CS38 in the absolute error value most often.

For mitigating of aliasing in DASP applications, the proposed nonuniform-based interpolation rules show enhanced results over uniform ones when Nyquist rate is not met.

REFERENCES

[1] A. Cauchy, "Mémoire sur diverses formules d'analyse," C. R. Acad. Sci., vol. 12, pp. 283–298, 1841.

[2] A. I. Zayed, "On sampling theorems and the Gauss-Jacobi mechanical quadrature", The Mathematical Heritage of C F Gauss, pp. 887-899, September 1991.

[3] C. E. Shannon, "Communication in the presence of noise," Proc. IRE, vol. 37, pp. 10–21, Jan. 1949.

[4] H. Nyquist, (April 1928) "Certain topics in telegraph transmission theory". Trans. AIEE. 47: 617–644. doi:10.1109/t-aiee.1928.5055024. Reprint as classic paper in: Proc. IEEE, Vol. 90, No. 2, Feb 2002

[5] F. Marvasti, Nonuniform Sampling, Theory and Practice. New York: Kluwer/Plenum, 2001.

[6] J. L. Yen, "On nonuniform sampling of bandwidth-limited signals," IRE Trans. Circuit Theory, vol.CT-3, pp. 251–257, Dec. 1956.

[7] J. L. Yen, "On the synthesis of line-sources and infinite strip-sources", 1957.

[8] H. S. Shapiro and R. A. Silverman, "Alias-free sampling of random noise," J. Soc. Ind. Appl. Math., vol. 8, no. 2, pp. 225–248, 1960.

[9] I. Bilinskis, Digital Alias-free Signal Processing, John Wiley & Sons, 2007.

[10] A. Tarczynski and B. I. Ahmad, "Estimation of Fourier Transform Using Alias-Free Hybrid-Stratified Sampling," in IEEE Transactions on Signal Processing, vol. 64, no. 12, pp. 3065-3076, June15, 2016.

[11] W. B. Ye and Y. J. Yu, "An efficient FIR filtering technique for processing non-uniformly sampled signal," in 2015 IEEE Inter. Conf. on Digital Signal Processing (DSP), pp. 182–186, 2015.

[12] H. Y. Darawsheh and A. Tarczynski, 2018. "Filtering Nonuniformly Sampled Grid-Based Signals." 2018 4th International Conference on Frontiers of Signal Processing (ICFSP 2018). Poitiers, France 24 - 27 Sep 2018 IEEE .

[13] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions: with formulas, graphs, and mathematical tables. Courier Corporation; 1964.