Optimal periodic sampling sequences for nearly-alias-free digital signal processing.

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***Abstract*—Alias-free DSP (DASP) is a methodology of processing signals digitally inside bandwidths that are wider than the famous Nyquist limit of half of the sampling frequency. DASP is facilitated by suitable combination of nonuniform sampling and appropriate processing algorithms. In this paper we propose a new method of constructing sampling schemes for the needs of DASP. Unlike traditional approaches that rely on randomly selected sampling instants we use deterministic schemes. A method of optimizing such sequences aimed at minimization of aliasing is proposed. The approach is tested numerically in an experiment where an undersampled signal is processed using DASP; first to estimate the signal's spectrum support function and then the spectrum itself. We demonstrate advantages of the proposed approach over those that use random sampling.**

I. INTRODUCTION

The constantly increasing power of digital equipment encourages researchers and engineers to use DSP at frequency ranges that are higher and wider than ever before. However, in order to use classical DSP in wide-band signal processing it is often necessary to increase the sampling rates to levels beyond the economically and / or technically viable limits. Today's fastest AD converters have bandwidths in the range of 2GHz. But the sampling rates they can achieve are up to 80 times lower than their doubled bandwidth see e.g. [1]. Despite this deficiency, such converters can still be used to acquiring data for many digital wide-band signal-processing tasks. However, in order to cope with undersampling suitable methodologies for collecting and processing data have to be deployed. Digital Alias-free Signal Processing (DASP) is an approach to tackle such issues. One of the earliest DASP-type ideas has been reported in [2]. In that paper Shapiro and Silverman have shown that by using suitable random sampling it is possible to estimate the power spectral density of stationary processes even though signal's bandwidth exceeds arbitrarily much half of the average sampling frequency. This important theoretical result has not found a direct way to practical applications because of at least two vital constraints. The sampling schemes used there require that occasionally the consecutive sampling instants are arbitrarily close to each other. Also, infinite number of samples is needed to resolve the problem of spectral analysis. Bilinskis and Mikelsons [3] have originated more practice-orientated DASP approaches. Reported applications of DASP include spectral analysis [4]-[6] and reconstruction of periodic signals [7]. Suitability of DASP for software radio applications has been investigated in [8]. Problems related to designing hardware of data acquisition systems for the needs of DASP are discussed in [9]

Traditionally, only *random* sampling schemes are used in DASP. Therefore, the quality of the results obtained from DASP is predictable in a statistical rather than deterministic sense [2]-[8]. Lack of bias proves that the constructed estimators are alias free. Standard deviations characterize their accuracy [5]-[6]. Such statistics cannot assure quality standards in each single experiment. In this paper we use *deterministic* sampling, namely Periodic Nonuniform Sampling (PNS) schemes. PNS is a useful tool for processing band-pass signals [10]. If the signal Spectrum Support Function (SSF) is known then PNS can be designed to keep signal aliases away from those frequency ranges that support the spectrum. Here we do not assume that, before processing starts, the signal's SSF is known with sufficient accuracy to achieve this effect. Therefore, our goal is to minimize the level of aliasing in the entire range of frequencies rather than to eliminate it completely but locally.

In the next section we overview the properties of PNS sampled signals. Then we introduce a formal measure of aliasing for such signals. In section IV we discuss how to optimize PNS to reduce the level of aliasing. The paper is concluded with a numerical example and final discussion.

II. PERIODIC NONUNIFORM SAMPLING

Consider a sampling sequence $\{\cdots, t_{-2}, t_{-1}, t_0, t_1, \cdots\}$. We define the sampling function associated with this sequence as a train of impulses

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - t_n) . \tag{1}$$

The work on this paper has been supported by IST-2001-34552 DASPTOOL project funded by the European Commission

The sampled signal is the product of the continuous time signal $x(t)$ and $s(t)$: $x_d(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x_n \delta(t - t_n)$, where $x_n = x(t_n)$. Our sampling scheme is PNS when $s(t)$ is periodic, i.e. there exists $T > 0$ such that for any t $s(t) = s(t + T)$. By using Fourier series we get

$$s(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T} t} . \qquad (2)$$

where $c_k = \frac{1}{T}\int_0^T s(t) e^{-j2\pi \frac{k}{T} t} dt = \frac{1}{T}\sum_{n=1}^{N} e^{-j2\pi \frac{k}{T} t_n}$ and N denotes the number of samples within one period *T*. It follows from (2) that the Fourier transform of $s(t)$ is given by

$$S(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T}\right) \qquad (3)$$

The spectrum $X_d(f)$ of the discrete-time signal is convolution of the spectrum $X(f)$ of the continuous-time signal and $S(f)$. Hence

$$X_d(f) = \sum_{k=-\infty}^{\infty} c_k X\left(f - \frac{k}{T}\right) \qquad (4)$$

The relation between the discrete- and continuous-time signals is thus fully defined by sequence $\{c_k\}_{k=-\infty}^{\infty}$. Note that $c_0 = N/T = f_s$. Here f_s is often called the (average) sampling frequency. If the SSF of $x(t)$ is unknown then, in order to avoid aliasing, coefficients c_k should satisfy $c_k = \delta_k f_s$. In such a case we could use $X(f) = X_d(f)/f_s$ to extract the spectrum of the continuous-time signal. Unfortunately, it is impossible to construct PNS producing so valued c_k. Therefore in this paper we focus on minimizing, rather than zeroing $|c_k|$ when $k \neq 0$.

III. Measuring Amount of Aliasing

In this section we introduce a quantitative measure of the level of aliasing. The method proposed here is best suited for signals consisting of narrow-band components.

Let $f_{\max} = k_{\max}/T$ be the highest frequency up to which we process signals. Consider a test signal $x(t) = \cos(2\pi f_0 t)$, whose frequency $f_0 \le f_{\max}$ is otherwise completely unknown. We determine which coefficients c_k in (2)-(4) ought to be minimized in order to reduce those aliases of $x(t)$ that appear in $[0,\ f_{\max}]$. According to (4) the spectrum of sampled $x(t)$ consists of a scaled copy of $X(f)$ (the fundamental alias): $X_{d0}(f) = 0.5 f_s(\delta(f - f_0) + \delta(f + f_0))$ and a combination of spurious components (other aliases of $X(f)$) that look like sinusoids of frequencies $f_0 \pm k/T$ and amplitudes $|c_k|$:

$$X_{d1}(f) = 0.5\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} c_k\left[\delta(f - f_0 - k/T) + \delta(f + f_0 - k/T)\right] \qquad (5)$$

To suppress unwanted aliases we should reduce the amplitudes $|c_k|$ of these components of $X_{d1}(f)$ whose frequencies $f_0 \pm k/T$ could appear for some value of f_0 in $[0,\ f_{\max}]$. Analysis of (5) shows that the indexes of such $|c_k|$ are $-2k_{\max} \le k \le 2k_{\max}$, with obvious exception $k \neq 0$. Since $|c_k| = |c_{-k}|$ it suffices to concentrate only on positive values of k. In this paper we choose the following index to measure the amount of aliasing

$$J = \max\left\{|c_1|/c_0,\quad |c_2|/c_0,\quad \cdots,\quad |c_{2k_{\max}}|/c_0\right\}. \qquad (6)$$

IV. Optimal Sampling Sequences

In order to ease practical implementation of the designed sampling scheme we require that the PNS not only minimizes (6) but also satisfies the following constraints: (a) all sampling instants are multiples of some time-interval L and (b) the distance between any two consecutive sampling instants is never shorter than *H*. Design of PNS sequences is now reduced to solving the following optimization problem.

Given: L, $H = rL$, $T = pL$ and $k_{\max}$, where r, p, $k_{\max}$ are integers. Determine the number N and positions of the sampling instants inside period T:

$$t_n = m_n L, \qquad n = 0, \cdots, N-1 \qquad (7)$$

so that (6) is minimized and integer numbers m_n satisfy the following constraints:

$$m_{n+1} \ge m_n + r \ \text{ for } \ n = 0, \cdots, N-1 \qquad (8)$$

$$m_{N-1} - m_0 + r \le p . \qquad (9)$$

We note that the set of feasible solutions, i.e. sampling sequences that satisfy constraints (7)-(9) but not necessarily minimize (6) is discrete and finite. Therefore the traditional optimization techniques such as gradient descent, conjugate direction, Newton method etc. that rely on continuity of the space are not directly applicable here. On the other hand, a systematic search through the space of feasible solutions guarantees finding the optimal one. The main disadvantage of such an approach is that in complex cases the number of feasible solutions is very large hence waiting time for the result could be long. The method we propose here is based on systemic search through a small subset of the set. Because of lack of space we present only the main ideas and justifications of the optimization algorithm.

First, we note that by shifting all the sampling instants by the same distance we do not change the magnitudes of c_k. Therefore the quality (6) of any two sequences that are shifted versions of each other is always the same. Consequently we can confine our search to sequences such that $t_0 = 0$. Let $d_1,...,d_N$ be the distances between consecutive sampling instants inside each period T. Using again the arguments about shifting the sampling sequence in time domain we note that by changing the order of those distances by circling them we do not affect the quality of the sampling sequence. In other words $d_1,...,d_N$ is the same good as $d_2,...,d_N,d_1$ or $d_3,...,d_N,d_1,d_2$ etc. These observations allow us to select a tiny subset of feasible solutions that needs to be searched in order to find a sampling sequence of the best achievable quality.

V. Numerical Example

Consider a narrow band signal concentrated about unknown frequency $f_c \in [0 \quad 120]$ MHz. Our task is to reconstruct the spectrum of the signal from its samples. Assume that the data acquisition board collecting the samples of the signal is built around an AD converter whose sampling rate cannot exceed $f_{\max} = 120$ MSps. See [1] for examples of such converters (e.g. MAX1190). In our case, if the samples were collected on a uniform grid, the bandwidth of DSP would not exceed 60MHz. In order to solve the stated problem we apply nonuniform sampling and DASP algorithms.

We start with designing PNS for collecting signal samples. To this end we choose the period T, step L and the distribution of the sampling instants inside a single period T. The minimum permissible distance between the sampling instants is determined by the maximum sampling rate of the converter: $H = 1/f_{\max} = 8.33 \times 10^{-9}$ s. We investigate four values of L: $L_1 = H/2$, $L_2 = H/3$, $L_3 = H/4$ and $L_4 = H/5$. In our search we vary the period of the sampling sequence between $T_{\min} = 5H$ and $T_{\max} = 20H$ in steps of L. Fig. 1 presents the results of search for the optimal PNS. The four lines, one for each value of L, show the smallest achievable cost (6) against the length of the period T. The best results were obtained when $L = H/5$ and $T = 20H$. Fig. 2 shows the sampling function associated with one period of the optimal sequence. We apply this sequence to a test signal $x(t) = 150\,\mathrm{sinc}(150 \times 10^6 t) - 130\,\mathrm{sinc}(130 \times 10^6 t)$. Its spectrum stretches from 65 to 75MHz. The magnitude of the spectrum is -120dB. The signal is observed and sampled over the interval of length $5T$. The collected samples are used to construct the preliminary approximation of the signal spectrum:

$$X(f) \approx \frac{1}{f_S} X_d(f) \approx \frac{1}{f_S} \sum_{n=1}^{n_{\max}} x(t_n) e^{-j2\pi f t_n} \qquad (10)$$

where $n_{\max} = 80$ is the number of collected samples. The thin line in Fig. 3 shows the magnitude of estimated spectrum (10). Although this type of spectral estimation is very crude it can provide enough information to approximate the signal's SSF. Using the thin line from Fig. 3 we conservatively estimated that the spectrum of the signal is placed between 60MHz and 80MHz. Based on this estimation we could calculate minimum energy reconstruction of the sampled signal [11] and its Fourier transform. The thick line in Figure 3 shows the magnitude of the estimated spectrum. This result is very accurate. The small ripples around -120dB level are the result of signal truncation rather than deficiency of the approach.

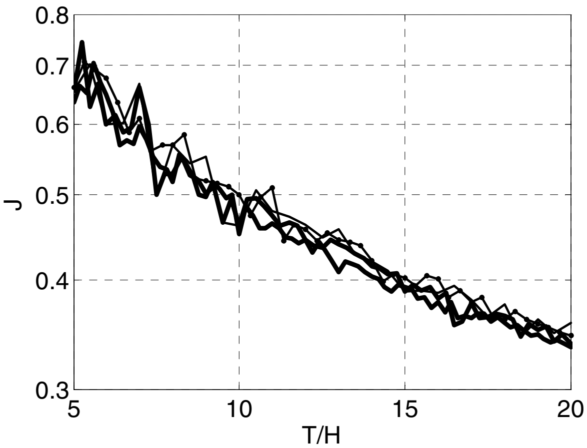

Figure 1. Cost (6) against length of period *T*: $L = H/2$ (thin dotted line), $L = H/3$ (thin continuous line), $L = H/4$ (thick dotted line) and $L = H/5$ (thick continuous line)

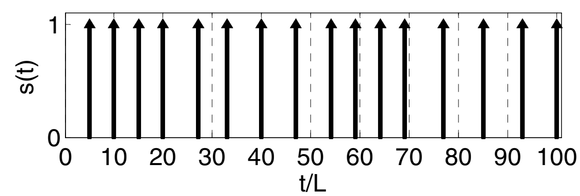

Figure 2. Optimal distribution of the sampling instants within one period

The role of the used sampling sequence in the above example is clear. It was designed in such a way that within the imposed constraints, all unwanted aliases of the signal were suppressed as much as possible. Consequently, despite use of low average sampling rate of 96 MHz the ratio between the fundamental and other aliases was large enough to allow sufficiently accurate estimation of SSF of the original signal. This information combined with good suppression of aliasing of the sampling scheme allowed us to accurately estimate the signal spectrum. Other sampling schemes could be less useful for solving this problem. Uniform sampling is definitely not an option here since the achievable rates are too slow to avoid aliasing. The outcomes of DASP techniques based on random sampling are more difficult to guess. A priori analysis cannot predict how good / bad could the results be in each experiment. Our tests have indicated that the quality can vary significantly from case to case. Figure 4 illustrates how bad the initial spectral analysis can go when additive random sampling [3] of $x(t)$ is used. In both presented cases it is much more difficult to correctly identify the signal's SSF than when optimized PNS is used. Wrong estimation of SSF at this stage could lead to poor-quality reconstruction of the signal and afterwards to erroneous estimation of its spectrum jeopardizing all the numerical efforts put in processing the signal.

VI. CONCLUSIONS

In this paper we have proposed using PNS schemes for the needs of DASP. We have shown how to design such sequences in order to maximize suppression of aliasing in the required range of frequencies. Optimization criterion (6) used here is best suited to preparing sampling schemes for signals consisting of narrow-band components whose widths do not exceed $1/T$ Hz. When the signal components occupy wider spaces in the frequency domain, then their neighboring aliases overlap each other and consequently criterion (6), though still useful, may not reflect in the best possible way the adverse effect of aliasing on the spectrum of the sampled signal. Better tackling of such cases is the subject of our further research.

The optimization techniques used for determining the best sampling sequence guarantee that, under the technical constraints explained in this paper, we always get the optimal PNS. The space that needs searching is a very small subset of the set of feasible solutions. Alternative methods that would further increase the speed of search but at the expense of possible loss of quality of the sampling sequence are currently investigated.

The quality of processing PNS data for various classes of input signals can be predicted by analyzing the spectrum of the associated sampling sequence and relation (4). Similar approach is not readily available for random sampling. Therefore for the DASP technique proposed here the quality assurance and guarantees can be much firmer than when more traditional approaches are deployed.

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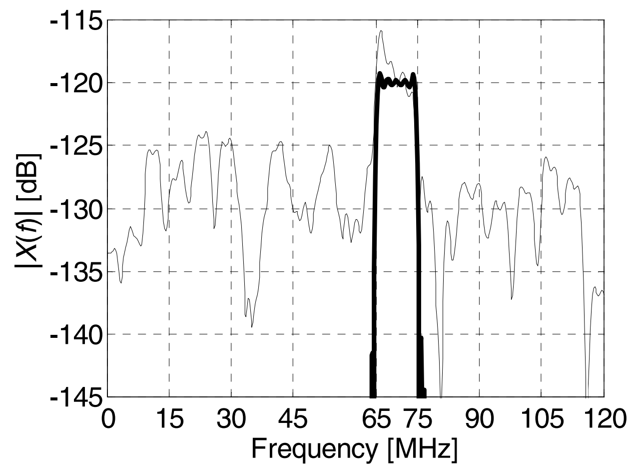

Figure 3. Magnitude of the signal spectrum estimated directly from the samples (thin line) and from minimum energy reconstruction (thick line).

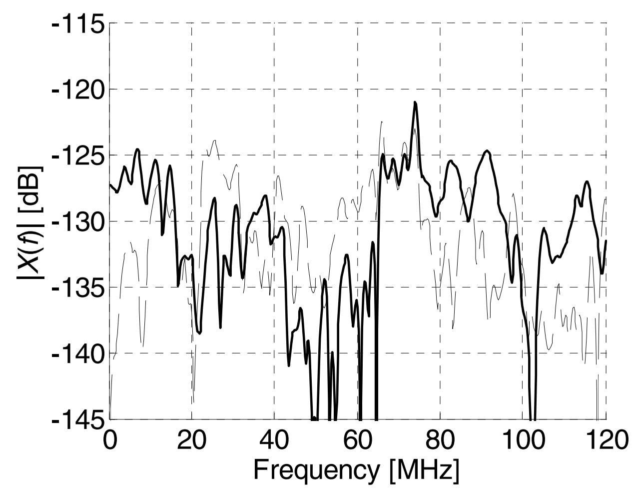

Figure 4. Magnitude of spectra of discrete-time signals obtained from two random sampling sequences.