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Downside Risk Measurement In Regime Switching Stochastic Volatility

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Abstract

Risk measurement is important to firms to enable management of risks, and ensure profitability during different firm and market events. In particular, downside risk is an important risk measure as it is a coherent risk measure, and it is also compatible with industry risk management approaches such as stop losses. Whilst regime switching models have been used for downside risk measurement, the regime switching models for stochastic volatility dynamics have been limited and so restrict risk measurement. In this paper we propose a new regime switching model that incorporates non-trivial stochastic volatility dynamics, hence we are able to measure risk more realistically. We derive the downside risk measure associated with our regime switching model, for risk measurement including and excluding jump risk. We prove that the regime switching model converges to the underlying continuous time asset pricing model, hence our risk measurement is consistent. We provide a discretisation for the variance risk process, which is locally consistent and enables computational implementation. We also provide numerical experiments to illustrate our method.

Keywords: downside risk; regime switching; stochastic volatility; jump risk; risk management.

AMS subject classifications: 91G80, 62P05, 97M30

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1 Introduction

Risk measurement is important to firms, as firms need to be able to manage their risks. The importance of risk measurement has particularly increased since the commencement of the Global Financial Crisis, where there were concerns of a global economic depression, and inappropriate risk measurement was considered a major cause of the Global Financial Crisis. One particularly relevant risk measure is downside risk, this is the expected loss in relation to some benchmark, or a numerical value. This risk measure is especially useful to risk management because firms are more concerned with quantifying losses in relation to risk, rather than quantifying gains, and downside risk achieves this objective. Additionally, many firms in industry enforce stop losses, where firms sell assets if prices fall below some benchmark value. Therefore the downside risk measure enables one to measure risk in an industry compatible approach. Furthermore, risk measures in general enable firms to minimise losses and maximise potential returns, determine their reserves, as well as inform hedging activities.

In order to measure the risk of assets, the traditional assumption of asset price modelling is model volatility as a constant, in particular one assumes a continuous time process of geometric Brownian motion. This implies asset returns are lognormally distributed and provides a simple and analytically tractable model. The geometric Brownian motion model possesses many useful theoretical properties, such as positive asset prices with probability 1, it enables the derivation and analysis of a number of important financial topics e.g. derivative pricing.

Despite the benefits of geometric Brownian motion, the process is considered unsuitable for asset price modelling for a number of reasons, in particular the assumption of constant volatility. There is significant empirical (and theoretical) evidence to suggest that asset models have non-constant volatility: firstly the implied volatility of option prices exhibit smiles (Renault and Touzi, 1996) whereas a constant volatility model would predict no smile in empirical option data. The return distribution of assets possess a fatter left tail and peakedness compared to the distribution expected for constant volatility models (Durham, 2007). Most notably, the stock market crash of 1987 empirically demonstrates a sudden and rapid change in volatility (a comprehensive study is given in (Schwert, 1990)) that cannot be explained by constant volatility models.

Consequently, correct risk modelling of assets must incorporate stochastic volatility to realistically capture the risk dynamics of asset prices, that is volatility that is a function of an additional random process. One method of stochastic volatility modelling is regime switching (or Markov chains). A regime switching model implies that a model with u_0 states, where $u_0 \in \mathbb{N}$, will switch between $\mathbb{N} \in \{1, 2, ..., u_0\}$ possible parameter values following some stochastic process. This is an attractive method of modelling stochastic volatility for a number of

reasons: firstly, it captures many of the empirical and theoretical properties of non-constant volatility that are important in risk modelling, such as extreme price movements that were exhibited during the Global Financial Crisis. Secondly, regime switching retains many of the analytical advantages of the geometric Brownian motion model, unlike many other stochastic volatility models (see for example (Heston, 1993) and (Hull and White, 1987)). Consequently, the regime switching model enables us to develop closed form solutions and computational methods.

In (Hardy, 2001) a stochastic volatility model is implemented using regime switching and the downside risk measure (also called the conditional tail expectation) is analytically derived with a closed form solution. The risk measure also has a maximum likelihood estimation method derived, and it is also shown that the model provides a significantly better fit than alternative models. However, there exist some areas for expansion in (Hardy, 2001): firstly, the regime switching model only incorporates a limited set of volatility dynamics, hence the risk measurement that can be captured by the model is restricted. Secondly, the relation of the regime switching model to the associated continuous time, stochastic differential equation model is not fully investigated. For example, how closely does the regime switching model relate to the underlying continuous time stochastic differential equation? To what extent does the regime switching model accurately capture the marginal distributions of the underlying stochastic process? Consequently, one cannot know the extent to which the regime switching model accurately allows risk measurement of the underlying process.

In this paper we propose a regime switching model that is able to model a wide range of stochastic volatility processes, including jumps and other non-trivial stochastic processes. In this paper we follow (Nguyen, 2018). We derive the downside risk measure for our regime switching model, for risk measurement including jump risk and excluding jump risk. We derive a discretised variance risk process, which has local consistency and enables computational implementation. We also prove that our regime switching model provides consistent risk measurement in that the model converges to the underlying continuous time stochastic differential equation.

The plan of the paper is as follows: first we begin with preliminaries and related literature. In the next section we introduce our underlying continuous time, stochastic differential equation model for the asset model, which includes a wide range of stochastic volatility processes. We then introduce our regime switching model associated with the underlying continuous time, stochastic differential equation model. In the next section we derive the downside risk measure for our regime switching model, without jump risk and including jump risk. In the next section we prove that our risk measure is consistent with risk measurement on the underlying asset model, as the number of regimes increase. Next, we conduct numerical experiments, analyse

the results and finally we end with a conclusion.

2 Preliminaries

Let us assume that a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ exists, where Ω denotes the sample space, $\mathcal{F}$ denotes a collection of events in Ω with probability measure $\mathbb{P}$, and we have a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$. We denote the set $\{\mathcal{F}_t\}$ as the set of information available to an individual up to time t, so that

$$\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}_T < \infty, \ \forall t_1 < t_2 < T.$$

We also denote the set $\{\mathcal{F}_t\}$, $t \in [0, T]$ as a filtration. Furthermore, for a given stochastic process $V(t)$, we define the filtration $\mathcal{F}_t^V$ to denote the information produced by $V(t)$ on the interval $[0, t]$, as more information is revealed to an individual as time t progresses. Finally, assume we have the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ then we define a change of measure $\mathbb{P} \sim \mathbb{Q}$ to be defined on the probability space $\{\Omega, \mathcal{F}, \mathbb{Q}\}$.

By Girsanov's Theorem with respect to stochastic differential equations and change of probability measures, let us assume we have a family of information sets $\mathcal{F}_t$ over a period $[0, T]$. We define over $[0, T]$ the random process (also known as the Doleans exponential) υ_t:

$$\upsilon_t = \exp\left\{-\int_0^t \hat{\gamma}(u) dB^{\mathbb{P}}(u) - \frac{1}{2}\int_0^t \hat{\gamma}^2(u) du\right\},$$

where $B^{\mathbb{P}}(t)$ is the Wiener process under probability measure $\mathbb{P}$ and $\hat{\gamma}(t)$ is an $\mathcal{F}_t$-measurable process that satisfies the Novikov condition

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^t \hat{\gamma}^2(u) du\right\}\right] < \infty, t \in [0, T].$$

Therefore $B^{\mathbb{Q}}(t)$ is defined as a Wiener process with respect to $\mathcal{F}_t$ under probability measure $\mathbb{Q}$, where $B^{\mathbb{Q}}(t)$ is given by

$$B^{\mathbb{Q}}(t) = B^{\mathbb{P}}(t) + \int_0^t \hat{\gamma}(u) du, t \in [0, T].$$

Let us define our asset pricing model as

$$dV(t)/V(t) = \alpha dt + \beta(\cdot) dB(t),$$

where $V(t)$ is the asset price at time t, $\alpha \in \mathbb{R}$ is the drift, and $\beta(\cdot)$ is the volatility process. The standard model for asset pricing assumes volatility as a constant (Black and Scholes, 1973), that is we have geometric Brownian motion

$$dV(t)/V(t) = \alpha dt + \beta dB(t).$$

Despite the analytical (and computational) advantages of geometric Brownian motion, the model is not considered sufficient for risk measurement due to constant volatility. Firstly, empirical option data frequently exhibits implied volatilities that are inconsistent with a constant volatility model (Renault and Touzi, 1996). Secondly, the return distributions that would be obtained for constant volatility models are not exhibited in empirical data (Durham, 2007). Finally, the stock market crash of 1987 clearly exhibited a large and rapid change in volatility that cannot be explained by constant volatility modelling (see for instance Bates (2018), Vo and Ellis (2018) and Schwert (1990)).

As a result of constant volatility models being unable to accurately model asset pricing dynamics, this has motivated new volatility models. The first development in non-constant volatility modelling was time dependent volatility modelling (see for example (Wilmott et al., 1998)):

$$dV(t)/V(t) = \alpha dt + \beta(t)dB(t).$$

We model volatility as a function of time t, and so volatility is no longer constant. In (Merton, 1973) derived the option pricing equation associated with this volatility model, using the standard Black-Scholes equation where volatility is replaced by β_c, where

$$\beta_c = \sqrt{\frac{1}{T-t}\int_t^T \beta^2(\tau)d\tau},$$

so that d_1 and d_2 in the Black-Scholes equation become:

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{V(t)}{K}\right) + \alpha(T-t) + \frac{1}{2}\int_t^T \beta^2(\tau)d\tau}{\sqrt{\int_t^T \beta^2(\tau)d\tau}}, \\ d_2 &= \frac{\log\left(\frac{V(t)}{K}\right) + \alpha(T-t) - \frac{1}{2}\int_t^T \beta^2(\tau)d\tau}{\sqrt{\int_t^T \beta^2(\tau)d\tau}}, \end{aligned}$$

where T denotes the option maturity, and K is the option strike price.

Local volatility modelling has been another method of non-constant volatility modelling, where the volatility is a function of stock price and time, that is $\beta = f(V(t), t)$. The Constant Elasticity of Variance model (CEV) is given by (Cox and Ross, 1976)

$$\begin{aligned} \frac{dV(t)}{V(t)} &= \alpha dt + \beta(V(t))dB(t), \\ \beta(V(t)) &= \tilde{\kappa}V^{\tilde{\eta}-1}(t), \ \textit{for} \ \{\tilde{\eta} \in \mathbb{R} | 0 \leq \tilde{\eta} \leq 1\}, \tilde{\kappa} \in \mathbb{R}^+. \end{aligned}$$

This model provides a flexible framework for modelling different volatility dynamics. For example, for $\tilde{\kappa} = 0$ we obtain Bachelier's model of stock prices, while $\tilde{\kappa} = 1$ gives the geometric

Brownian motion model. The CEV model is also popular in industry and so has been extended over time (for instance (Cox and Ross, 1976), (Cox et al., 1985) and (Schroder, 1989)), and provides parsimonious calibration to option data (see (Beckers, 1980) for more information).

Dupire's local volatility model (Dupire, 1994) has Dupire's equation, which is obtained by applying the Fokker-Planck equation in terms of a call option $H(t)$:

$$\frac{\partial H(t)}{\partial T} = \beta^2(V(t),T).\frac{V^2(t)}{2}.\frac{\partial^2 H(t)}{\partial V^2(t)} - (r-D)V(t).\frac{\partial H(t)}{\partial V(t)} - D.H(t), \quad (1)$$

where D is the asset dividend. It can be shown that from equation (1) we can derive

$$\beta(V(t),T) = \sqrt{\frac{\frac{\partial H(t)}{\partial T} + (r-D)V(t)\frac{\partial H(t)}{\partial V} + D.H(t)}{\frac{V^2(t)}{2}.\frac{\partial^2 H(t)}{\partial V^2(t)}}}. \quad (2)$$

Hence equation (2) implies volatility $\beta(V(t),T)$ can be extracted from option data, however this requires partial derivatives with respect to T and K. Consequently, we require a continuous set of options data in K and T, and this is typically not available without excluding high transaction costs(Nordén, 2003).

The most comprehensive development in volatility modelling has been stochastic volatility, which can take into account empirical properties such as the clustering effect, fatter tail distributions, implied volatility smiles and time scaling effects (Musiela and Rutkowski, 2005). The stochastic volatility model is given by

$$\begin{aligned} dV(t)/V(t) &= \alpha_1(V(t),t)dt + \beta(t)dB^1(t), \\ \beta(t) &= f(dB^2(t)), \end{aligned}$$

where volatility is a function of a stochastic process that is driven by another (but possibly correlated) Wiener process $dB^2(t)$, specified by correlation constant ρ where $\rho \in \{\mathbb{R}|-1 \leq \rho \leq 1\}$. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is $\Omega = \mathcal{C}([0,\infty) : \mathbb{R}^2)$, with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ to represent information on two Wiener processes $\{B^1(t), B^2(t)\}$.

One of the first stochastic volatility models is (Johnson and Shanno, 1987), where

$$d\beta(t) = \alpha_2\beta(t)dt + \beta^{\tilde{\kappa}}(t)\tilde{\eta}dB^2(t), \ for \ \{\tilde{\kappa},\tilde{\eta} \in \mathbb{R}|\tilde{\kappa},\tilde{\eta} \geq 0\}.$$

No analytical solution is provided for option prices in (Johnson and Shanno, 1987) , although option prices are determined by Monte Carlo methods. Another stochastic volatility applies the Ornstein-Uhlenbeck process (Scott, 1987):

$$d\beta(t) = \alpha_2(\tilde{\kappa} - \alpha(t))dt + \tilde{\eta}dB^2(t), \ for \ \{\tilde{\kappa},\tilde{\eta} \in \mathbb{R}|\tilde{\kappa},\tilde{\eta} \geq 0\}.$$

The Hull-White Model (Hull and White, 1987) is an alternative stochastic volatility model:

$$d\beta^2(t)/\beta^2(t) = \alpha_2 dt + \tilde{\eta}dB^2(t), \ for \ \{\tilde{\kappa},\tilde{\eta} \in \mathbb{R}|\tilde{\kappa},\tilde{\eta} \geq 0\},$$

and one can obtain option prices using the Black-Scholes option pricing equation, with volatility $\hat{\beta}^2$ where

$$\hat{\beta}^2 = \frac{1}{\tilde{T}-t}\int_t^{\tilde{T}} \beta^2(s)ds.$$

The Heston stochastic volatility model (Heston, 1993) takes in account correlation between Wiener processes $corr(dB^1(t), dB^2(t)) = \rho dt$, where the volatility process is given by

$$d\beta^2(t) \;=\; \alpha_2(\tilde{\kappa} - \beta^2(t))dt + \tilde{\eta}\beta(t)dB^2(t),\ for\ \{\tilde{\kappa}, \tilde{\eta} \in \mathbb{R} | \tilde{\kappa}, \tilde{\eta} \geq 0\}. \tag{3}$$

Whilst the models on non-constant volatility have been developed, separately the work on risk measurement has been developed by (Artzner et al., 1999), which undertook an axiomatic approach. The fundamental work of (Artzner et al., 1999) defines coherent risk measurement, that is axioms that risk measures should obey in order to measure risk correctly, based on the sample space of losses. If we assume we have a real valued random variable $\mathcal{Y} \in \mathbb{R}$ within the measurable space $\{\Omega, \mathcal{F}\}$, where $\mathcal{Y}$ follows a distribution of losses $\mathcal{G}$, then a risk measure $\Phi(\cdot)$ is defined by

$$\Phi(\mathcal{Y}) : \mathcal{G} \mapsto \mathbb{R}.$$

A risk measure is considered coherent (that is risk is measured correctly) if the risk measure abides to the coherency axioms (Artzner et al., 2003). This axiomatic approach to risk measurement has proven beneficial to specifying risk measurement. The coherency axioms are translation invariance, subaddivity, monotonicity and positive homogeneity and are given (respectively) as

$$\begin{aligned}
\Phi(\mathcal{Y} + k) &= \Phi(\mathcal{Y}) + k,\ \text{for } k \in \mathbb{R}, \\
\Phi(\mathcal{Y}_1 + \mathcal{Y}_2) &\leq \Phi(\mathcal{Y}_1) + \Phi(\mathcal{Y}_2), \\
\Phi(\mathcal{Y}_1) &\leq \Phi(\mathcal{Y}_2), \forall \mathcal{Y}_1 \leq \mathcal{Y}_2, \\
\Phi(k\mathcal{Y}) &= k\Phi(\mathcal{Y}), \forall k \in \mathbb{R}_{\geq 0}.
\end{aligned}$$

For the purposes of risk measurement we will assume risk is measured under the risk neutral probability measure unless stated otherwise; this does not change any of the results but is used for convenience and ease of comparsion.

The translation invariance axiom ensures that a cash position (reflected by a constant) has no impact on risk, since cash is riskless. The subadditivity axiom implies that a portfolio has less risk than the sum of the risk of the individual assets. This axiom takes into account diversification in portfolios, and is also a criterion that disqualifies many risk measures as coherent, for example VaR (Value at Risk). The monotonicity axiom implies that riskier assets should have higher risk values, and the scale invariance implies that the magnitude of investment does not affect the level of risk itself in the asset.

There exist a large number of coherent (and non-coherent) risk measures that have been studied in many papers (see for example (Szegö, 2005)). One of the most popular risk measures in research and industry is Value at Risk (VaR), which is defined as

$$VaR(\mathcal{Y}) = \mathbb{F}(\mathcal{Y} \leq k)$$

where $\mathbb{F}(\cdot)$ is the cumulative distribution function. Essentially, VaR specifies a quantile at a given cumulative probability; this is typically $90\%, 95\%$, and 99%. The VaR risk measure benefits from analytical tractability, parsimonious computational implementation, and the risk measure is popular with many managers as it is simple to comprehend. These advantages outweigh some of the criticisms of VaR, such as incorrect risk measurement of diversified portfolios.

Other popular risk measures include variance (or equivalently standard deviation) and statistical moments, that is we have the risk measure

$$\Phi(\mathcal{Y}) = \mathbb{E}[\mathcal{Y}^n],$$

or in other words the n^{th} moment, where $\mathbb{E}[.]$ is the expectation. The moment measure of risk is a convenient measure, and it is well-known that moments give useful information about the distribution of random variables. In fact in (Hoyland and Wallace, 2001) the moments are used to produce scenarios for optimisation modelling.

One other class of risk measures are the upside and downside risk measures, and such risk measures are coherent (Szegö, 2005) hence they will measure risk correctly. The upside risk measure is given by

$$\Phi(\mathcal{Y}) = \mathbb{E}[(\mathcal{Y} - k)^+],$$

and the downside risk is given by

$$\Phi(\mathcal{Y}) = \mathbb{E}[(k - \mathcal{Y})^+],$$

where $k \in \mathbb{R}$ is some constant. The upside risk measure enables us to gauge the expected gain of $\mathcal{Y}$ beyond some threshold value k, whereas downside risk enables us to measure the loss in $\mathcal{Y}$ below some threshold value k.

The downside risk measure is popular in industry because many firms are interested in determining their expected performance, relative to some benchmark or constant k. For example, firms are frequently interested in outperforming a stock market index, or some other benchmark of performance. Hence the downside risk measure is able to quantify the risk of underperforming such a benchmark. Secondly, the downside risk measure is able to incorporate aspects of Behavioural Finance theory, hence the risk measure is more suited to actual investor behaviour. In particular, the downside risk measure incorporates Prospect Theory, whereby investors are sensitive to losses beyond some behavioural "reference point". The risk measure is able to take into account such a reference point in terms of benchmark k.

3 Asset Model

The regime switching model is associated with an underlying continuous time, stochastic differential equation that models the asset prices. Additionally, we want to model non-trivial price processes of assets in order to realistically capture the risk of assets, in particularly we want to include non-trivial stochastic volatility processes. Let us assume that our asset price $V(t)$ follows the process

$$dV(t) = \alpha(\cdot)V(t)dt + \beta(.)V(t)dB^1(t) + V(t)(e^{\Lambda(t)} - 1)dN(t), \tag{4}$$

where $\alpha(\cdot)$ is the drift, $B^1(t)$ is a Brownian motion, $\Lambda(t) = \sum_{n=1}^{N(t)} Z(n)$ with $Z(n)$ for n=1,2,.. is a sequence of i.i.d. (independent and identically distributed) random variables, each with the same density function $f_\Lambda(z)$ that defines the jump sizes. The $N(t)$ represents a Poisson process, with jump amplitude $\Lambda(t)$, with rate $\lambda > 0$.

As mentioned previously, a key issue in the risk modelling of assets is the modelling of the volatility process $\beta(.)$, given that volatility modelling is an important component in asset price moves. As the purpose of the asset model is to capture risk, we specify our model to be a function of some parsimonious risk process. One approach is to use the Markowitz model (Markowitz, 1952) as it is one of the most widely used measures of risk in industry. In (Markowitz, 1952) there are n assets in a portfolio with return $\mathbf{r(t)}$ where

$$\begin{aligned} \mathbf{r(t)} &= [r_1(t), r_2(t), ..., r_n(t)]^T, \\ \mathbb{E}[\mathbf{r(t)}] &= \mu, \end{aligned}$$

with covariance matrix

$$\Sigma = \mathbb{E}[(\mathbf{r(t)} - \mu)(\mathbf{r(t)} - \mu)^T],$$

and assets weights $\mathbf{w} = [w_1, ..., w_n]^T$. The purpose of (Markowitz, 1952) is to minimise

$$arg \min_{\mathbf{w}} \mathbb{E}[\mathbf{w}^T\mu - \mathbf{w}^T\mathbf{r(t)}], \tag{5}$$

where the portfolio has expected return $\mathbf{w}^T\mu$ and variance $\mathbf{w}^T\Sigma\mathbf{w}$. Therefore the risk process in (Markowitz, 1952) is modelled by the variance of assets or portfolios.

As variance is an important risk process in (Markowitz, 1952), in order to capture risk in our model we also model volatility as a function of the variance process $\vartheta(t)$, that is $\beta = f(\vartheta)$, similarly the drift is also a function of $\vartheta(t)$, that is $\alpha = f(\vartheta(t))$. Consequently our model becomes

$$dV(t) = \alpha(\vartheta(t))V(t)dt + \beta(\vartheta(t))V(t)dB^1(t) + V(t)(e^{\Lambda(t)} - 1)dN(t).$$

Moreover, to enable us to model a wide range of risk processes, we assume our variance process $\vartheta(t)$ follows the stochastic differential equation

$$d\vartheta(t) = \hat{\eta}(\vartheta(t))dt + \hat{\sigma}(\vartheta(t))dB^2(t),$$

where $\hat{\sigma}(\vartheta(t)) > 0$ is a Brownian motion coefficient. The $B^1(t)$ and $B^2(t)$ are correlated Brownian motions, that is $\mathbb{E}[dB^1(t)dB^2(t)] = \rho dt$, and $\{\rho \in \mathbb{R} | -1 \leq \rho \leq 1\}$, the $N(t)$ represents a Poisson process, with jump amplitude $\Lambda(t)$, with rate $\lambda > 0$, and is independent of either Brownian motions $B^1(t), B^2(t)$. We now have a non-trivial and comprehensive asset pricing model, which can take into account many realistic features of asset prices, in particular stochastic jumps, stochastic volatility and correlated Brownian motions.

We apply Ito's Lemma to $log(V(t))$ so that we have

$$\begin{aligned} d(log(V(t)) &= \frac{1}{V(t)}d\bar{V}(t) - \frac{1}{2V^2(t)}(d\bar{V}(t))^2 + d\sum_{0<s\leq t}[log(V(s)) - log(V(s^-)] \\ &= \left(\alpha(\vartheta(t)) - \frac{1}{2}\beta^2((\vartheta(t))\right)dt + \beta(\vartheta(t))dB^1(t) + d\Lambda(t), \end{aligned}$$

where $\bar{V}(t)$ is the continuous function component of $V(t)$. Let us now define

$$\begin{aligned} \hat{j}(x) &= \int_k^x \frac{\beta(s)}{\sigma(\hat{s})}ds, \\ \Gamma(x) &= \mathcal{L}(\hat{j}(x)) = \hat{\eta}(x)\hat{j}'(x) + \frac{1}{2}\hat{\sigma}^2(x)\hat{j}''(x), \\ j(\vartheta(t), \vartheta(0)) &= \rho(\hat{j}(\vartheta(t)) - \hat{j}(\vartheta(0))), \end{aligned}$$

where k is a constant. We then have

$$dj(\vartheta(t), \vartheta(0)) = \rho d\hat{j}(\vartheta(t)) = \rho\Gamma(\vartheta(t))dt + \rho\beta(\vartheta(t))dB^2(t).$$

Now let us set

$$B^*(t) = \frac{(B^1(t) - \rho B^2(t))}{\sqrt{1-\rho^2}},$$

then $\mathbb{E}[dB^*(t)dB^2(t)] = 0$, where $B^*(t)$ is a standard Brownian motion. Hence $B^*(t)$ and $B^2(t)$ are statistically independent Brownian motions.

If we now substitute the previous equation into our equation for $d(log(V(t))$ this gives

$$\begin{aligned} d(log(V(t)) &= \left(\alpha(\vartheta(t)) - \frac{1}{2}\beta^2((\vartheta(t))\right)dt + d\hat{j}(\vartheta(t), \vartheta(0)) - \rho\Gamma(\vartheta(t))dt \\ &+ \sqrt{1-\rho^2}\beta(\vartheta(t))dB^*(t) + d\Lambda(t). \end{aligned}$$

Let us also introduce $\tilde{S}(t) = log\left(\frac{V(t)}{V(0)}\right) - j(\vartheta(t), \vartheta(0))$, then we have
$V(t) = V(0)exp(\tilde{S}(t) + j(\vartheta(t), \vartheta(0)))$. Moreover we can express

$$d\tilde{S}(t) = \left(\alpha(\vartheta(t)) - \frac{1}{2}\beta^2(\vartheta(t)) - \rho\Gamma(\vartheta(t))\right)dt + \sqrt{1-\rho^2 t}\beta(\vartheta(t))dB^*(t) + d\Lambda(t), \quad (6)$$

$$d\vartheta(t) = \hat{\eta}(\vartheta(t))dt + \hat{\sigma}(\vartheta(t))dB^2(t). \quad (7)$$

We now have an equation for the asset price process, where the variance process $d\vartheta(t)$ is expressed in terms of an independent Brownian motion $B^2(t)$ from $B^*(t)$. Additionally, this equation will enable the regime switching risk analysis to be more easily derived.

We now wish to introduce our regime switching model related to our continuous time asset pricing model. The regime switching model itself was introduced by Hamilton (see for instance (Hamilton, 1989), (Hamilton, 1991), (Hamilton and Susmel, 1994), (Hamilton, 1994)) who proposed that economic cycles follow regime switching processes. The objective of such models is typically focussed towards econometric modelling, and so have been combined with ARCH-type models, but our focus is on risk measurement. The rationale for economic time series following regime switching processes are highly consistent with economic and financial theory. For example, economic processes tend to exhibit memory or "clustering effects" (see for example (Engle, 1982)), hence high volatility events tend to be preceded by previous high volatility events. Additionally, there exist statistical dependencies on switching from one "mode" to another "mode"; in both cases regime switching models are able to incorporate such attributes.

We define our regime switching model as a u_0-state, continuous time, Markov chain, where $u_0 \in \mathbb{N}^+$. We also have the random process $\phi(t)$ that determines the state, so that $\phi(t) \in \{1, 2, ..., u_0\}$ and the transition between states is specified by the generator matrix $\mathcal{Q}$. Given that $\vartheta(\cdot)$ is our risk variable we define $\vartheta(\cdot)$ as a function of $\phi(t)$, thus we discretise $\vartheta(t)$ into space $\{\vartheta(1), \vartheta(2), ..., \vartheta(u_0)\}$. Hence we have a u_0-state, continuous time, Markov chain $\vartheta(\phi(t))$, where $\phi(t) \in \{1, 2, ..., u_0\}$, with transition between states specified by the generator matrix $\mathcal{Q}$. We can now approximate $\tilde{S}(t)$ by a regime switching jump diffusion process:

$$dS(t) = \eta(\phi(t))dt + \sigma(\phi(t))dB^*(t) + d\Lambda(t), \tag{8}$$

where

$$\begin{aligned} \eta(t) &= \alpha(\vartheta(\phi(t))) - \frac{1}{2}\beta^2(\vartheta(\phi(t))) - \rho\Gamma(\vartheta(\eta(t))), \\ \sigma(\phi(t)) &= \sqrt{1-\rho^2}\beta(\vartheta(\phi(t))). \end{aligned}$$

It has been demonstrated in many papers that simple regime switching models are sufficiently flexible to model a wide variety of asset price processes. For example, (Bollen, 1998) utilises regime switching models to value American options, in (Mamon and Rodrigo, 2005) regimes model the economy to price options and obtain analytic solutions, and in (Zhou and Mamon, 2011) regime switching models are applied to model interest rate dynamics. Consequently, the introduction of regime switching does not limit the modelling ability of the model.

4 Downside Risk Measurement

In order to derive risk measures for our regime switching model, we will construct a lattice. In our lattice construction we will restrict our trees to recombining trees, with 3 branches for each node: an up move in the asset price, a down move in the asset price, and a no-price move (denoted by the middle branch in the tree). We consider the finite time interval of interest for risk analysis $[0, T]$, where $T > 0$, and divide $[0, T]$ into $N \in \mathbb{N}$ subintervals of length, that is $\varkappa := \frac{T}{N}$. We now consider our model and measure the downside risk, with and without jump risk.

4.1 Risk Measurement Without Jump Risk

We first consider our model without jump risk, that is

$$dS(t) = \eta(\phi(t))dt + \sigma(\phi(t))dB^*(t). \tag{9}$$

It is beneficial to consider our model without jump risk (also known as event risk) because jump risk itself represents a significant and separate risk component of the model. In fact the presence of jumps causes Black-Scholes replicating portfolios to contain unhedged risk (see (Merton, 1976)) that is a function of the trading intervals. The jump components are considered to model new and firm specific information on an asset (Merton, 1976) and the arrival of such information follows a random process. Consequently, we would like to examine the model with and without jump risk to have a better understanding of the risk.

Let us denote $(S(k), \phi(k)) := (S(t), \phi(t))_{t=k\varkappa}$ to give the approximated state at the k^{th} step of the tree; let us also assume initially that $(S(k), \phi(k)) = (s, i)$ at time step k. Let $\bar{\sigma} > 0$ denote the space step size for variable S; let us denote the number of upward lattice movements of $S(k+1)$ by κ_i, where $\kappa_i \in \mathbb{N}^+$. Let us also denote by $p_{i,u}, p_{i,d}, p_{i,c}$ the lattice branch probabilities of upwards move, downwards move and no move (or middle move), respectively, where

$$\begin{aligned} p_{i,u} + p_{i,d} + p_{i,c} &= 1, \text{ where} \\ 0 \leq p_{i,u} &\leq 1, \\ 0 \leq p_{i,d} &\leq 1, \\ 0 \leq p_{i,c} &\leq 1, \end{aligned}$$

by law of total probability. We define the probabilities as

$$p_{ij} = P(\phi_{k+1} = j | \phi_k = i), \forall i, j \in \{1, 2, ..., u_0\},$$

therefore the three possible lattice branch values for $S(k+1)$ are: $s + \kappa_i \bar{\sigma}\sqrt{\varkappa}$ (for upwards move), $s - \kappa_i \bar{\sigma}\sqrt{\varkappa}$ (for downward move), and s (for no or middle branch move). As the purpose of the

model is to enable risk measurement, we apply a condition to ensure our lattice price moves are meaningful in terms of risk and using equation (9). Consequently, we use the Markowitz risk metric (that is equation (5)) and so we match the mean and variance implied by our model in equation (9), to the mean and variance of our lattice. Hence we obtain

$$p_{i,u} = \frac{\sigma_i^2 + \eta_i(\kappa_i\bar{\sigma}\sqrt{\varkappa}) + \eta_i^2\varkappa}{2(\kappa_i\bar{\sigma})^2}, \tag{10}$$

$$p_{i,d} = \frac{\sigma_i^2 - \eta_i(\kappa_i\bar{\sigma}\sqrt{\varkappa}) + \eta_i^2\varkappa}{2(\kappa_i\bar{\sigma})^2}, \tag{11}$$

$$p_{i,c} = 1 - \frac{\sigma_i^2 + \eta_i^2\varkappa}{(\kappa_i\bar{\sigma})^2}. \tag{12}$$

Our resultant lattice gives the following structure, starting from node (s, i) at step k, there will be $3u_0$ possible nodes for $(S(k+1), \phi(k+1))$ at step $(k+1)$. The $(S(k+1), \phi(k+1))$ nodes $\forall j \in \{1, 2, .., u_0\}$ are given by:

$$(s + \kappa_i\bar{\sigma}\sqrt{\varkappa}, j) \text{ with probability } p_{ij}p_{i,u}, \tag{13}$$

$$(s - \kappa_i\bar{\sigma}\sqrt{\varkappa}, j) \text{ with probability } p_{ij}p_{i,d}, \tag{14}$$

$$(s, j) \text{ with probability } p_{ij}p_{i,c}. \tag{15}$$

In terms of lattice price moves, in order for the lattice to provide meaningful asset price movements we impose the following constraint:

$$(\kappa_i\bar{\sigma})^2 > \sigma_i^2 + \eta_i^2\varkappa.$$

This constraint again is determined in terms of providing meaningful movements in terms of risk. The lattice movement $(\kappa_i\bar{\sigma})^2$ should exceed asset movement due to the drift movement in the asset (that is $\eta_i^2\varkappa$) plus movements associated with risk (that is we add σ_i^2). We note we measure risk for each branch by squaring σ_i, hence all other terms are squared to provide an appropriate scaling.

Now in order for our lattice model to be a viable model, we require branch probabilities $p_{i,c} \in [0, 1], p_{i,u} \in [0, 1], p_{i,d} \in [0, 1]$. This can be achieved with the following Lemma.

Lemma 1.

For the model

$$dS(t) = \eta(\phi(t))dt + \sigma(\phi(t))dB^*(t),$$

the branch probabilities in the lattice model are necessarily bound within the following intervals $p_{i,c} \in [0, 1], p_{i,u} \in [0, 1], p_{i,d} \in [0, 1]$ for $i \in \{1, 2, ..., u_0\}$, $\eta_i \neq 0$, if

$$0 < \varkappa \leq \min_{1 \leq i \leq u_0} \frac{(\kappa_i\bar{\sigma})^2 - \sigma_i^2}{\eta_i^2}, \text{ where } 0 < \sigma_i < \kappa_i\bar{\sigma} \leq 2\sigma_i.$$

Proof.

We first determine the inequalities for $p_{i,c}$: let us first prove $p_{i,c} \geq 0$. From equation (10) we have

$$\begin{aligned} p_{i,c} = 1 - \frac{\sigma_i^2 + \eta_i^2 \varkappa}{(\kappa_i \bar{\sigma})^2} \Rightarrow 1 - p_{i,c} &= \frac{\sigma_i^2 + \eta_i^2 \varkappa}{(\kappa_i \bar{\sigma})^2}, \\ (1 - p_{i,c})(\kappa_i \bar{\sigma})^2 &= \sigma_i^2 + \eta_i^2 \varkappa, \\ \varkappa &= \frac{(1 - p_{i,c})(\kappa_i \bar{\sigma})^2 - \sigma_i^2}{\eta_i^2}. \end{aligned}$$

Also, given that

$$\varkappa \leq \min_{1 \leq i \leq u_0} \frac{(\kappa_i \bar{\sigma})^2 - \sigma_i^2}{\eta_i^2} \Rightarrow p_{i,c} \geq 0.$$

We now prove $p_{i,c} \leq 1$. From equation (10) we have

$$p_{i,c} = 1 - \frac{\sigma_i^2 + \eta_i^2 \varkappa}{(\kappa_i \bar{\sigma})^2}.$$

In order for $p_{i,c} \leq 1$ we require

$$\begin{aligned} (\kappa_i \bar{\sigma})^2 &\geq \sigma_i^2 + \eta_i^2 \varkappa, \\ \varkappa &\leq \frac{(\kappa_i \bar{\sigma})^2 - \sigma_i^2}{\eta_i^2}. \end{aligned}$$

This condition is met from the inequality

$$0 < \varkappa \leq \min_{1 \leq i \leq u_0} \frac{(\kappa_i \bar{\sigma})^2 - \sigma_i^2}{\eta_i^2}, \text{ where } 0 < \sigma_i < \kappa_i \bar{\sigma} \leq 2\sigma_i.$$

Hence we can conclude from the two inequalities that $p_{i,c} \in [0, 1]$.

Now let us determine the inequalities for $p_{i,u}$: from equation (10) we have

$$p_{i,u} = \frac{\sigma_i^2 + \eta_i (\kappa_i \bar{\sigma} \sqrt{\varkappa}) + \eta_i^2 \varkappa}{2(\kappa_i \bar{\sigma})^2},$$

thus to ensure $p_{i,u} \geq 0$ we require

$$\begin{aligned} \eta_i^2 \varkappa + \eta_i \kappa_i \bar{\sigma} \sqrt{\varkappa} + \sigma_i^2 &\geq \eta_i^2 \left[-\frac{(\kappa_i \bar{\sigma})}{2\eta_i} \right]^2 + \eta_i \kappa_i \bar{\sigma} \left[-\frac{(\kappa_i \bar{\sigma})}{2\eta_i} \right] + \sigma_i^2, \\ &\geq \sigma_i^2 - \frac{(\kappa_i \bar{\sigma})^2}{4}. \end{aligned}$$

Therefore we have $p_{i,u} \geq 0$.

We now prove the condition for $p_{i,u} \leq 1$: we have the inequality

$$\begin{aligned} 0 < \varkappa \leq \min_{1 \leq i \leq u_0} \frac{(\kappa_i \bar{\sigma})^2 - \sigma_i^2}{\eta_i^2} &\Rightarrow \eta_i^2 \varkappa \leq (\kappa_i \bar{\sigma})^2 - \sigma_i^2 \\ &\Rightarrow \eta_i^2 \varkappa - (\kappa_i \bar{\sigma})^2 + \sigma_i^2 \leq 0. \end{aligned}$$

We also have $\eta_i\kappa_i\bar{\sigma}\sqrt{\varkappa} \leq (\kappa_i\bar{\sigma})^2 \Rightarrow \eta_i\kappa_i\bar{\sigma}\sqrt{\varkappa} - (\kappa_i\bar{\sigma})^2 \leq 0$. If we now combine both inequalities we have

$$\begin{aligned} \eta_i^2\varkappa + \eta_i\kappa_i\bar{\sigma}\sqrt{\varkappa} + \sigma_i^2 - 2(\kappa_i\bar{\sigma})^2 &\leq 0, \\ \eta_i^2\varkappa + \eta_i\kappa_i\bar{\sigma}\sqrt{\varkappa} + \sigma_i^2 &\leq 2(\kappa_i\bar{\sigma})^2. \end{aligned}$$

Thus from equation (10)

$$p_{i,u} = \frac{\sigma_i^2 + \eta_i(\kappa_i\bar{\sigma}\sqrt{\varkappa}) + \eta_i^2\varkappa}{2(\kappa_i\bar{\sigma})^2},$$

we have $p_{i,u} \leq 1$. If we now combine both inequalities then we have $0 \leq p_{i,u} \leq 1$. By a similar argument it can be shown $0 \leq p_{i,d} \leq 1$. This completes our proof. ■

We now derive our downside risk measurement for our regime switching model. Assume we wish to measure risk between time 0 and T, with threshold Υ. Let $X_\Upsilon^k(s,i)$ denote downside risk measurement at time step k, for node with state $(S(k),\phi(k)) = (s,i)$, with final time step $k = N$, we have

$$X_\Upsilon^k(s,i) = (\Upsilon - V(0)exp(s)]^+, \text{ for } i = 1,2,...,u_0.$$

Now the downside risk at time $k, \forall k \in [1, N-1]$ can be calculated *recursively*, hence this provides a significant efficiency in computation. The recursive calculation is

$$\begin{aligned} X_\Upsilon^k(s,i) &= \sum_{j=1}^{u_0} p_{ij}(p_{i,u}X_\Upsilon^{k+1}(s+\kappa_i\bar{\sigma}\sqrt{\varkappa},j) + p_{i,c}X_\Upsilon^{k+1}(s,j) && (16) \\ &+ p_{ij}(p_{i,u}X_\Upsilon^{k+1}(s-\kappa_i\bar{\sigma}\sqrt{\varkappa},j)). && (17) \end{aligned}$$

4.2 Risk Measurement with Jump Risk

We now wish to include jump risk in our model for risk measurement; to construct our lattice for equation (8) whilst incorporating jumps we apply (Amin, 1993). Therefore, recalling that $(S(k),\phi(k)) = (s,i)$ at time step k, the Poisson process with rate $\lambda > 0$, on the the interval $[k\varkappa, ((k+1)\varkappa))$, we have the probabilities: for a single jump given by $\lambda\varkappa + \mathcal{O}(h)$, for more than one jump given by $\mathcal{O}(h)$, and for no jump is given by $1 - \lambda\varkappa + \mathcal{O}(h)$. We also assume by choosing sufficiently small $\varkappa$ that multiple jumps do not occur within $[k\varkappa, ((k+1)\varkappa))$. Now with respect to the jump size $Z(k)$, with probability density function $f_\Lambda(z)$, the cumulative distribution function for $Z(k)$ is given by

$$F(x) = \int_{-\infty}^{x} f_\Lambda(z)dy,$$

the probability density function is approximated using a discretisation $\{\kappa\bar{\sigma}\sqrt{\varkappa}, \kappa = 0, \pm1, \pm2, ...\}$. If we denote the approximated discrete jump size by $\bar{Z}(k)$, then the associated probability mass

function for $\bar{Z}(k), \forall\kappa = 0, \pm1, \pm2, ...,$ is given by

$$P(\bar{Z}(k) = \kappa\bar{\sigma}\sqrt{\varkappa}) = dF(\kappa) = F((\kappa + 0.5)\bar{\sigma}\varkappa) - F((\kappa - 0.5)\bar{\sigma}\varkappa).$$

For our lattice model, given $(S(k), \phi(k)) = (s, i)$, for $1 \leq j \leq u_0$ the next value $(S(k + 1), \phi(k + 1))$ at the $(k + 1)$ step is given by

$$(S(k+1), \phi(k+1)) \quad = \quad (x + \kappa\bar{\sigma}\sqrt{\varkappa}, j), \tag{18}$$

with probability

$$p_{ij}[(1 - \lambda\varkappa)p_{i,u} + \lambda\varkappa dF(\kappa_i)],$$

for $\kappa \neq -\kappa_i, 0, \kappa_i$, otherwise with probability

$$p_{ij}[(1 - \lambda\varkappa)p_{i,u} + \lambda\varkappa dF(\kappa_i)].$$

For the other branches in the lattice model we have

$$\begin{aligned}
&(x, \kappa) \text{ with probability } p_{ij}[(1 - \lambda\kappa)p_{i,c} + \lambda\kappa dF(0)], \\
&(x - \kappa\bar{\sigma}\sqrt{\varkappa}, j) \text{ with probability } p_{ij}[(1 - \lambda\kappa)p_{i,d} + \lambda\varkappa dF(-\kappa_i)].
\end{aligned}$$

Now let $\kappa_u, \kappa_d, \kappa_c \in \{-\kappa_i, 0, \kappa_i\}$ and if we now use this model for downside risk measurement (taking into account jumps) then our recursive method (equation (16)) becomes

$$\begin{aligned}
X_\Upsilon^k(s, i) \quad &= \quad \sum_{j=1}^{u_0} p_{ij} \left[\sum_{r=u,c,d} [p_{i,r}(1 - \lambda\varkappa) + \lambda\varkappa dF(\kappa_r)] X_\Upsilon^{k+1}(s + \kappa_r\bar{\sigma}\sqrt{\kappa}, j) \right] \\
&+ \quad \sum_{j=1}^{u_0} p_{ij} \left[\sum_{\kappa\neq\kappa_u,\kappa_m,\kappa_d} \lambda\varkappa dF(\kappa) X_\Upsilon^{k+1}(s + \kappa\bar{\sigma}\sqrt{\varkappa}, j) \right].
\end{aligned}$$

5 Variance Risk Process

The variance risk process was previously specified to be discretised so that it follows a Markov chain, that is we have $\vartheta(\phi(t))$. We would like $\vartheta(\phi(t))$ to have local consistency and provide a meaningful discretisation in terms of risk, rather than an arbitrary discretisation. Consequently, we now define $\vartheta(\phi(t))$ more specifically using (Lo and Skindilias, 2014), where we can approximate the $\vartheta(\phi(t))$ process by a locally consistent continuous time Markov chain $\vartheta(\phi(t))$, with a state space $\vartheta = \{\vartheta(1), \vartheta(2), ..., \vartheta(u_0)\}$, such that $\vartheta(i - 1) < \vartheta(i)$. We note that we require the associated generator matrix $\mathcal{Q} := [q_{ij}]_{u_0\times u_0}$, where

$$\begin{aligned}
\sum_{j=0}^{u_0} q_{ij} \quad &= \quad 0, \forall i = 1, 2, ..., u_0, \\
q_{ii} \quad &\leq \quad 0, \forall i = 1, 2, ..., u_0.
\end{aligned}$$

Let $k_i = \vartheta_i - \vartheta_{i-1}$ for $i = 2, ..., u_0$, and $\mathbf{k} = \{k_1, k_2, ..., k_{u_0-1}\}$ then we specify $\mathcal{Q}$ elements as

$$\begin{aligned} q_{ij} &= \frac{\hat{\eta}^-(\vartheta(i))}{k_{i-1}} + \frac{\hat{\sigma}^2(\vartheta_i) - (k_{i-1}\hat{\eta}^-(\vartheta_i) + \hat{\eta}^+(\vartheta_i))}{k_{i-1}(k_{i-1}+k_i)}, \text{ for } j = i-1, \\ &= \frac{\hat{\eta}^+(\vartheta(i))}{k_{i-1}} + \frac{\hat{\sigma}^2(\vartheta_i) - (k_{i-1}\hat{\eta}^-(\vartheta_i) + \hat{\eta}^+(\vartheta_i))}{k_{i-1}(k_{i-1}+k_i)}, \text{ for } j = i+1, \\ &= -q_{i,i-1} - q_{i,i+1}, \text{ for } i = j, \\ &= 0, \text{ for } j \neq \{i-1, i, i+1\}. \end{aligned}$$

where $x^{\pm} = \max(0, \pm x)$.

The equations for q_{ij} imply that q_{ij} values may not be well specified (for instance $q_{ij} \geq 0$), in fact for q_{ij} to be well specified we require

$$\hat{\sigma}^2(\vartheta_i) \geq k_{i-1}\hat{\eta}^-(\vartheta_i) + \hat{\eta}^+(\vartheta_i).$$

Hence we must specify $k_i, \forall i$. Trivially we can impose $k_i > 0, \forall i$, however this is not a useful condition in terms of risk analysis; we want k_i to provide meaningful steps in the change in the state space, rather than an arbitrary choice of steps. The Sharpe ratio is a frequently used ratio in risk analysis to determine if meaningful asset gains are being achieved for a given level of risk. Hence we impose the following bound based on the optimal Sharpe ratio, so we have

$$\max_{1 \leq i \leq u_0 - 1} \frac{|\hat{\eta}| - r}{\vartheta_i} = \max_{1 \leq i \leq u_0 - 1} \frac{|\hat{\eta}|}{\vartheta_i}.$$

The maximisation removes r since r is a constant. We can alternatively express this as

$$\min_{1 \leq i \leq u_0 - 1} \frac{\vartheta_i}{|\hat{\eta}|}.$$

Consequently we impose the condition for choosing k_i

$$0 < \max_{1 \leq i \leq u_0 - 1} k_i \leq \min_{1 \leq i \leq u_0} \frac{\hat{\sigma}^2(\vartheta_i)}{|\hat{\eta}(\vartheta_i)|}.$$

Therefore

$$\begin{aligned} \hat{\sigma}^2(\vartheta_i) &\geq (\max_{1 \leq i \leq u_0 - 1} k_i)|\hat{\eta}(\vartheta_i)| \\ &\geq k_{i-1}\hat{\eta}^-(\vartheta_i) + k_i\hat{\eta}^+(\vartheta_i). \end{aligned}$$

So we have a well defined q_{ij}, that is $q_{ij} \geq 0$ for $1 \leq i \neq j \leq u_0$, and $\sum_{h=1}^{m_0} q_{ij} = 0, \forall i = 1, 2, ..., u_0$. Furthermore from (Lo and Skindilias, 2014) we can deduce

$$\begin{aligned} \mathbb{E}[\varkappa(t + \Delta t) - \varkappa(t)] &= \hat{\eta}(\vartheta)\varkappa, \\ \mathbb{E}[\varkappa(t + \Delta t) - \varkappa(t)]^2 &= (\hat{\sigma}(\vartheta))^2\varkappa. \end{aligned}$$

Consequently it can be shown ((Kushner, 1990)) that the Markov chain satisfies local consistency conditions.

To enable computational implementation of the variance process we require a grid specification. Consequently, we apply a method based on finite difference methods in finance (see for instance (Brandimarte, 2002), (Wilmott and Howison, 1995), (Wilmott et al., 1998)). In finite difference methods there is a significant literature on grid discretisation, we apply a finite difference grid that employs coordinate transformation. For example, for the standard Black-Scholes partial differential equation for the lognormal asset price process

$$\frac{\partial C}{\partial t} + \frac{\sigma_c^2}{2} V^2 \frac{\partial^2 C}{\partial V^2} + (r - D) V \frac{\partial C}{\partial t} - rC = 0,$$

can also be transformed with $V = V(\xi)$ so that we have the partial differential equation in terms of the new coordinate ξ, that is

$$\frac{\partial C}{\partial t} + \frac{\sigma_c^2}{2} . \frac{V^2(\xi)}{J(\xi)} . \frac{\partial}{\partial \xi} \left(\frac{1}{J(\xi)} \frac{\partial C}{\partial \xi} \right) + (r - D) . \frac{V(\xi)}{J(\xi)} . \frac{\partial C}{\partial \xi} - rC = 0,$$

where we use the Jacobian of transformation, that is

$$J(\xi) = \frac{dV(\xi)}{d\xi}.$$

Whilst this method is a standard approach to coordinate transformation, it would be more beneficial to concentrate grid points near critical values. In fact many coordinate transformation methods exist for such approaches (see for example (Tavella and Randall, 2000)), we apply the following standard transformation (Tavella and Randall, 2000):

$$J(\xi) = A(E^2 + (\mathcal{V}(\xi) - \mathcal{B})^2)^{\frac{1}{2}},$$

where $\mathcal{V}(\xi)$ is the transformed variable, A is a constant, E is a uniformity constant (to be explained later), and $\mathcal{B}$ is the region of interest. If we have the boundary conditions $\mathcal{V}(\xi = 0) = \mathcal{V}_{min}$ and $\mathcal{V}(\xi = 1) = \mathcal{V}_{max}$, where $\mathcal{V}_{min}$ and $\mathcal{V}_{max}$ are the minimum and maximum values of $\mathcal{V}$, respectively, then by integration we deduce

$$\mathcal{V}(\xi) = \mathcal{B} + E sinh(\varrho_2 \xi + \varrho_1 (1 - \xi)),$$

where

$$\begin{aligned} \varrho_1 &= sinh^{-1} \left(\frac{\mathcal{V}_{min} - \mathcal{B}}{E} \right), \\ \varrho_2 &= sinh^{-1} \left(\frac{\mathcal{V}_{max} - \mathcal{B}}{E} \right). \end{aligned}$$

We now apply this coordinate transformation method to our model. We set $\mathcal{V} = \vartheta(i) \Rightarrow \mathcal{V}_{min} = \vartheta(1), \mathcal{V}_{max} = \vartheta(u_0)$; our region of interest is $\vartheta(0)$ thus $\mathcal{B} = \vartheta(0)$, and we set $\xi = \frac{i}{u_0}$. For E is the uniformity constant, where uniformity in the grid increases with E. We set $E = \mathcal{A}$ and $\mathcal{A} = \vartheta(u_0) - \vartheta(1)$, we can then determine $\vartheta(i), \forall i = 2, 3, ..., u_0 - 1$ by

$$\vartheta(i) = \vartheta(0) + \mathcal{A} sinh \left(\varrho_2 \frac{i}{u_0} + \varrho_1 \left(1 - \frac{i}{u_0} \right) \right),$$

where

$$\varrho_1 = sinh^{-1}\left(\frac{\vartheta(1)-\vartheta(0)}{\mathcal{A}}\right), \varrho_2 = sinh^{-1}\left(\frac{\vartheta(u_0)-\vartheta(0)}{\mathcal{A}}\right).$$

In order to apply our grid to computation we are required to calculate ϱ_1, ϱ_2, therefore we need $\vartheta(1)$ and $\vartheta(u_0)$. We want to determine $\vartheta(u_0)$: as we measure risk within our branches in terms of variance, therefore we want $\vartheta(u_0)$ to be at least equal to its expected value plus increase in value due to risk, hence we set $\vartheta(u_0) = \zeta(t) + \alpha\mathcal{S}(t)$, where α is between 3-4, $\zeta(t) = \mathbb{E}[\vartheta(t)|\vartheta(0)]$, with $t = T/2$ and $\mathcal{S}(t)$ the standard deviation of $\vartheta(t)$ conditional on $\vartheta(0)$. For $\vartheta(1)$, given that $\vartheta(u_0) = \zeta(t) + \alpha\mathcal{S}(t)$ we would also want an equidistant difference from $\zeta(t)$, hence we could assign $\vartheta(1) = \zeta(t) - \alpha\mathcal{S}(t)$, however this leads to potentially small values for $\vartheta(1)$. Thus to ensure a sufficiently large $\vartheta(1)$ values we assign $\vartheta(1) = max(\bar{\vartheta}, \zeta(t) - \alpha\mathcal{S}(t))$, where $\bar{\vartheta} \approx 10^{-5}$.

6 Risk Measurement Consistency

In order for our model to provide consistent risk measurement we require that equation (8) converges to the underlying process (equation(6)). In particular we require convergence as $u_0 \to \infty$, otherwise our model will never provide correct risk measurement. This is because such a convergence implies convergence in the marginal distributions (to be discussed in the proceeding sections), consequently we will have convergence in risk measurement (since risk measurement is determined by the distribution properties of random variables).

To prove that equation (8) converges to the underlying process (equation(6)), as $u_0 \to \infty$, we require the following Lemma.

Lemma 2.

Let $d\tilde{S}^*(t)$ be the diffusion process

$$\begin{aligned} d\tilde{S}^*(t) &= \left(\alpha(\vartheta(t)) - \frac{1}{2}\beta^2(\vartheta(t)) - \rho\Gamma(\vartheta(t))\right)dt + \sqrt{1-\rho^2 t}\beta(\vartheta(t))dB^*(t), \\ d\vartheta(t) &= \hat{\eta}(\vartheta(t))dt + \hat{\sigma}(\vartheta(t))dB^2(t), \end{aligned}$$

and let $dS^*(t)$ be the process

$$dS^*(t) = \eta(\phi(t))dt + \sigma(\phi(t))dB^*(t).$$

Then $(S^*(t), \vartheta(\phi(t)))$ converges weakly to $(\tilde{S}(t), \vartheta(\phi(t)))$ if $\mathcal{Q} = q_{ij}$ is specified by

$$\begin{aligned} q_{ij} &= \frac{\hat{\eta}^-(\vartheta(i))}{k_{i-1}} + \frac{\hat{\sigma}^2(\vartheta_i) - (k_{i-1}\hat{\eta}^-(\vartheta_i) + \hat{\eta}^+(\vartheta_i))}{k_{i-1}(k_{i-1}+k_i)}, \text{ for } j = i-1, \\ &= \frac{\hat{\eta}^+(\vartheta(i))}{k_{i-1}} + \frac{\hat{\sigma}^2(\vartheta_i) - (k_{i-1}\hat{\eta}^-(\vartheta_i) + \hat{\eta}^+(\vartheta_i))}{k_{i-1}(k_{i-1}+k_i)}, \text{ for } j = i+1, \\ &= -q_{i,i-1} - q_{i,i+1}, \text{ for } i = j, \\ &= 0, \text{ for } j \neq i-1, i, i+1. \end{aligned}$$

Proof.

Let us prove $(S^*(t), \vartheta(\phi(t)))$ converges weakly to $(\tilde{S}(t), \vartheta(\phi(t)))$: firstly let us define $\varsigma(\vartheta(t)) := (\alpha(\vartheta(t)) - \frac{1}{2}\beta^2(\vartheta(t)) - \rho\Gamma(\vartheta(t)))$, and set $\tilde{S}(0) = s$, $\vartheta(0) = \vartheta$, and we also recall that $B^*(t)$ and $B^2(t)$ are independent Brownian motions. Now let $\mathbb{V} = \{\vartheta(1), \vartheta(2), ..., \vartheta(u_0)\}$, $\mathbb{S}$ is the state space of $\tilde{S}^*(t)$, $||f||_B = \sup_{x\in B} |f(x)|$, and let us define

$$E^{u_0} = ||\mathcal{L}X(s,\vartheta) - \mathcal{L}^{u_0}X(s,\vartheta)||_{\mathbb{S}\times\mathbb{X}},$$

where the infinitessimal generators $\mathcal{L}(.)$ are defined by

$$\mathcal{L}X(s,\vartheta) = \frac{1}{2}(1-\rho^2)[\beta(\vartheta)]^2\frac{\partial^2 X}{\partial s^2} + \varsigma(\vartheta)\frac{\partial X}{\partial s} + \frac{1}{2}\hat{\sigma}^2(\vartheta)\frac{\partial^2 X}{\partial \vartheta^2} + \hat{\eta}(\vartheta)\frac{\partial X}{\partial \vartheta}.$$

Also let $S^*(0) = s$, $\phi(0) = i$ and the generator for $(\tilde{S}^*(t), \vartheta(t))$ is given by

$$\begin{aligned}\mathcal{L}^{u_0}X(s,\vartheta(i)) &= \frac{1}{2}(1-\rho^2)[\beta(\vartheta)]^2\frac{\partial^2 X}{\partial s^2}(s,\vartheta(i)) + \varsigma(\vartheta(i))\frac{X}{s}(s,\vartheta(i)) \\ &+ \sum_j q_{ij}X(s,\vartheta(j)).\end{aligned}$$

By (Mijatovic and Pistorius, 2011) then we can conclude $(S^*(t), \vartheta(\phi(t)))$ converges weakly to $(\tilde{S}^*(t), \vartheta(\phi(t)))$, if $E^{u_0}(X) \to 0$, as $\Delta\vartheta := max_i\{|\vartheta(i) - \vartheta(i-1)\} \to 0$, or equivalently $u_0 \to \infty$.

To prove the convergence, we firstly use $\mathcal{Q}$ so that

$$\sum_j q_{ij}X(s,\vartheta(j)) = q_{i,i-1}X(s,\vartheta(i-1)) + q_{i,i}X(s,\vartheta(i)) + q_{i,i+1}X(s,\vartheta(i+1)),$$

therefore

$$\begin{aligned}&\mathcal{L}^{u_0}X(s,\vartheta(i)) \\ =\ & \frac{1}{2}(1-\rho^2)[\beta(\vartheta)]^2\frac{\partial^2 X}{\partial s^2}(s,\vartheta(i)) + \varsigma(\vartheta(i))\frac{X}{s}(s,\vartheta(i)) \\ +\ & \tilde{\eta}(\vartheta(i))\left[\frac{-k_i}{k_{i-1}(k_{i-1}+k_i)}X(s,\vartheta(i-1)) + \frac{k_i - k_{i-1}}{k_i k_{i-1}}X(s,\vartheta(i)) + \frac{-k_{i-1}}{k_i(k_{i-1}+k_i)}X(s,\vartheta(i+1))\right] \\ +\ & \frac{\hat{\sigma}^2}{2}\left[\frac{2}{k_{i-1}(k_{i-1}+k_i)}X(s,\vartheta(i-1)) - \frac{2}{k_{i-1}k_i}X(s,\vartheta(i)) + \frac{2}{k_i(k_{i-1}+k_i)}X(s,\vartheta(i+1))\right].\end{aligned}$$

This is a finite discretisation, and for convenience we use the notation

$$\nabla_\vartheta X(s,\vartheta(i)) = \frac{-k_i}{k_{i-1}(k_{i-1}+k_i)}X(s,\vartheta(i-1)) + \frac{k_i - k_{i-1}}{k_i k_{i-1}}X(s,\vartheta(i)) + \frac{-k_{i-1}}{k_i(k_{i-1}+k_i)}X(s,\vartheta(i+1)),$$

and

$$\nabla^2_\vartheta X(s,\vartheta(i)) = \frac{2}{k_{i-1}(k_{i-1}+k_i)}X(s,\vartheta(i-1)) - \frac{2}{k_{i-1}k_i}X(s,\vartheta(i)) + \frac{2}{k_i(k_{i-1}+k_i)}X(s,\vartheta(i+1)).$$

So we have

$$\hat{\eta}(\vartheta)\frac{\partial X}{\partial \vartheta}(s,\vartheta) + \frac{1}{2}\hat{\sigma}^2(\varsigma)\frac{\partial^2 X}{\partial \vartheta^2}(s,\vartheta),$$

having a finite discretisation with a grid $\{k_i = \vartheta_{i+1} - \vartheta_i : i = 1, 2, ..., u_0 - 1\}$:

$$\hat{\eta}(\vartheta(i))\nabla_\vartheta X(s, \vartheta(i)) + \frac{\hat{\sigma}^2(\vartheta(i))}{2}\nabla^2_\vartheta X(s, \vartheta(i)).$$

Therefore we have

$$\begin{aligned} E^{u_0}(X) &\leq ||\hat{\eta}||_{[\vartheta(1),\vartheta(u_0)]} \left|\left|\frac{\partial X}{\partial \vartheta} - \nabla_\vartheta X\right|\right|_{\mathbb{S}\times\mathbb{X}} + \frac{1}{2}||\hat{\sigma}^2||_{[\vartheta(1),\vartheta(u_0)]} \left|\left|\frac{\partial^2 X}{\partial \vartheta^2} - \nabla^2_\vartheta X\right|\right|_{\mathbb{S}\times\mathbb{X}} \quad (19) \\ \therefore E^{u_0}(X) &\to 0, \text{ as } \max_i\{|\vartheta(i) - \vartheta(i-1)\} \to 0. \quad (20) \end{aligned}$$

Hence by (Mijatovic and Pistorius, 2011) we can conclude $(S^*(t), \vartheta(\phi(t)))$ converges weakly to $(\tilde{S}^*(t), \vartheta(\phi(t)))$.

Now let $\theta(\chi) = \mathbb{E}[exp(i\chi Z_1)]$, where Z_1 follows a Normal distribution, that is $Z_1 \sim N(a, b)$. As we have independence of the jump component so

$$\begin{aligned} \mathbb{E}[exp(iS(t)\chi + i\vartheta(\phi(t))\varpi)] &= exp(t\lambda(\theta(\chi) - 1))\mathbb{E}[exp(iS^*(t)\chi + i\vartheta(\phi(t))\varpi)] \\ &\to exp(t\lambda(\theta(\chi) - 1))\mathbb{E}[exp(i\tilde{S}^*(t)\chi + i\vartheta(t))\varpi)]. \end{aligned}$$

As $exp(t\lambda(\theta(\chi) - 1))\mathbb{E}[exp(i\tilde{S}^*(t)\chi + i\vartheta(t))\varpi)] = \mathbb{E}[exp(i\tilde{S}(t)\chi + i\vartheta(t))\varpi)]$, therefore

$$\mathbb{E}[exp(iS(t)\chi + i\vartheta(\phi(t))\varpi)] \to exp(t\lambda(\theta(\chi) - 1))\mathbb{E}[exp(i\tilde{S}(t)\chi + i\vartheta(t))\varpi)].$$

Hence $(S(t), \vartheta(\phi(t)))$ weakly converges to $(\tilde{S}(t), \vartheta(\phi(t)))$. This completes the proof. ■

7 Numerical Experiments

In this section we conduct numerical experiments to illustrate our method for downside risk measurement. We test against 3 portfolios with initial value $\{\$18,000, \$18,500, \$19,000\}$, across a range of loss threshold values L in the range $L \in \{-\$5000, \$4000\}$, where a negative threshold value indicates a portfolio gain in value. We set the time period of risk measurement to $T = 0.25$ or 3 months to provide sufficient time period for a wider range of future random values to be realised. We also provide downside risk measurement results using a Monte Carlo simulation approach, to provide a comparison for results as well as enabling error analysis.

In terms of the regime switching model employed for our numerical experiments, we provide the following asset specification. Firstly, we require sufficient Markov states u_0 to ensure that we can model different volatility levels in the financial market, or alternatively we want to choose u_0 such that we are able to model a wide range of price dynamics. In Luo et al. (2019) they model a wide range of asset dynamics, for a number of different financial assets, using 30 regimes and so we set $u_0 = 30$ regimes. In terms of choice of time steps, given that $T = 0.25$ this implies we have the equivalent of 60 trading days or approximately 480 trading hours (each month has approximately 20 trading days, and each trading day lasts for approximately

8 hours). Additionally, we would not expect significant asset price moves in time intervals less than 30 minutes (on average) based on empirical data, hence we set time steps to 1000 for our numerical experiments (approximately equivalent to the number of 30 minute intervals in $T = 0.25$) to adequately capture all asset pricing dynamics.

The Heston stochastic volatility model is utilised (see equation (3)) for modelling assets. We choose the Heston model as it is able to model a wide range of asset pricing dynamics (such as stochastic volatility and mean reversion) and so provides theoretical and empirical consistent modelling advantages. In terms of industry and practical applications, the Heston model is a beneficial model to apply because it is popular in the financial industry (see for instance Feng and Wang (2018)and Zhang et al. (2016)). For example, the Heston model can enable pricing of call options that can be consistent with market observations and data.

The Heston model is set to be consistent with market calibrated figures. We note that the calibrated parameter values are not necessary for our analysis, however we explain them for completeness. Firstly we set asset drift $\alpha_1 = 0.1$, as stock market returns are on average approximately 10% per year, therefore 10% provides a realistic return in the stock market for any asset. The correlation of Wiener processes $corr(dB^1(t), dB^2(t)) = \rho dt$ tends to be weakly and positively correlated in financial markets, hence we set $\rho = 0.1$ and this is also consistent with Ikonen and Toivanen (2009). The parameter $\bar{\kappa}$ influences the long run value of β^2, and α_2 is the rate of mean reversion of β^2. As we would like sufficient volatility to ensure a wide range of price dynamics in our model we therefore set $\bar{\kappa} = 0.16$ and $\alpha_2 = 5$. We impose the Feller condition (see for instance Gatheral and Jacquier (2011) for more information) so that β is not negative, thus $2\alpha_2\bar{\kappa} \geq \bar{\eta}^2$ and therefore we set $\bar{\eta} = 0.9$.

We now present our results.

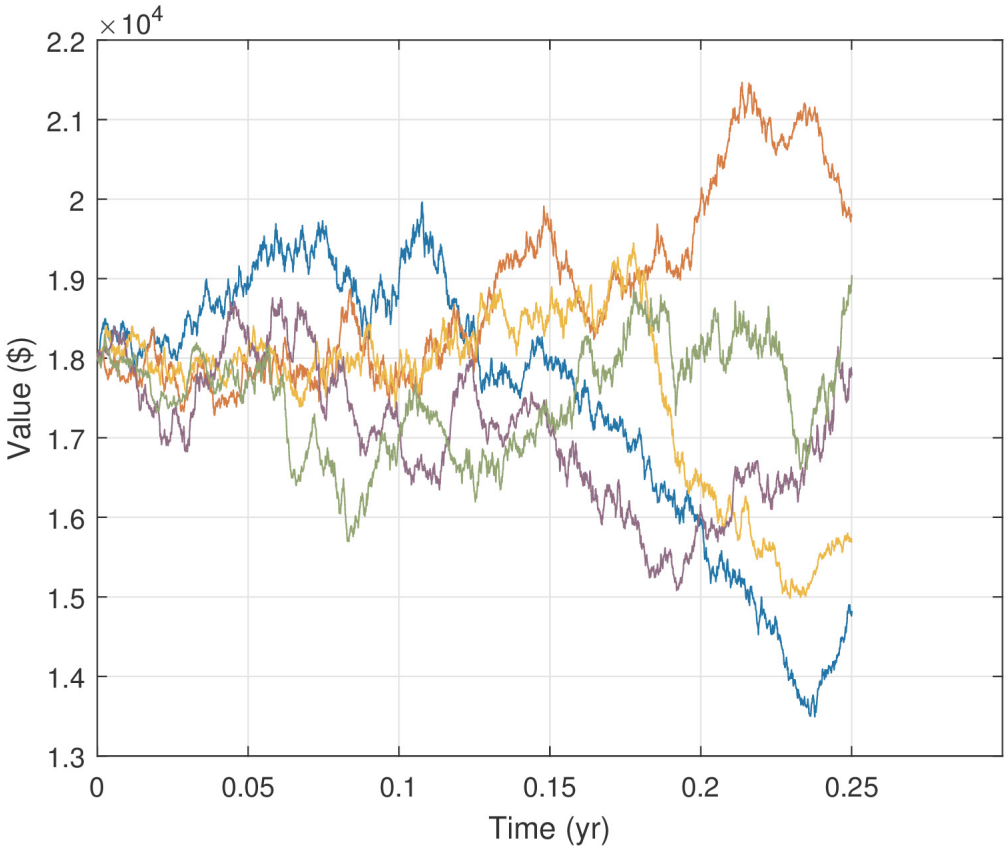

Figure 1: Graph of Sample Paths For Portfolio Value Over T=0.25 yr (With Initial Value $18,000)

Loss Threshold L ($)	**Downside Risk (MC)**	**Downside Risk (RS)**	**Absolute Error**
-4500	103.44	103.46	-0.0169
-2000	52.42	52.33	0.0850
0	22.57	22.43	0.1350
1600	8.52	8.40	0.1145
3000	3.02	2.94	0.0750

Table 1: Downside Risk Values for Different Loss Threshold Values L (Initial Portfolio Value $18,000)

Risk Measure	**Risk Value ($)**
Variance	8.9495×10^6
Sharpe Ratio	0.0068
μ_n, $n = 1$	-435.04
μ_n, $n = 2$	9137800
μ_n, $n = 3$	-3.3637×10^{10}
μ_n, $n = 4$	4.44×10^{14}
VaR_q, $q = 90\%$	3061.8
VaR_q, $q = 95\%$	3912.6
VaR_q, $q = 98\%$	4815.3
VaR_q, $q = 99\%$	5367.1

Table 2: Additional Risk Measures For Portfolio With Initial Value $18,000

Loss Threshold L ($)	**Downside Risk (MC)**	**Downside Risk (RS)**	**Absolute Error**
-4600	106.31	106.33	-0.0173
-2000	53.87	53.78	0.0874
0	23.19	23.06	0.1387
1600	8.75	8.64	0.1177
3000	3.10	3.02	0.0771

Table 3: Downside Risk Values for Different Loss Threshold Values L (Initial Portfolio Value $18,500)

Loss Threshold L ($)	**Downside Risk (MC)**	**Downside Risk (RS)**	**Absolute Error**
-4700	109.18	109.20	-0.0178
-2100	55.33	55.24	0.0897
0	23.82	23.68	0.1425
1700	8.99	8.87	0.1209
3100	3.18	3.10	0.0792

Table 4: Downside Risk Values for Different Loss Threshold Values L (Initial Portfolio Value $19,000)

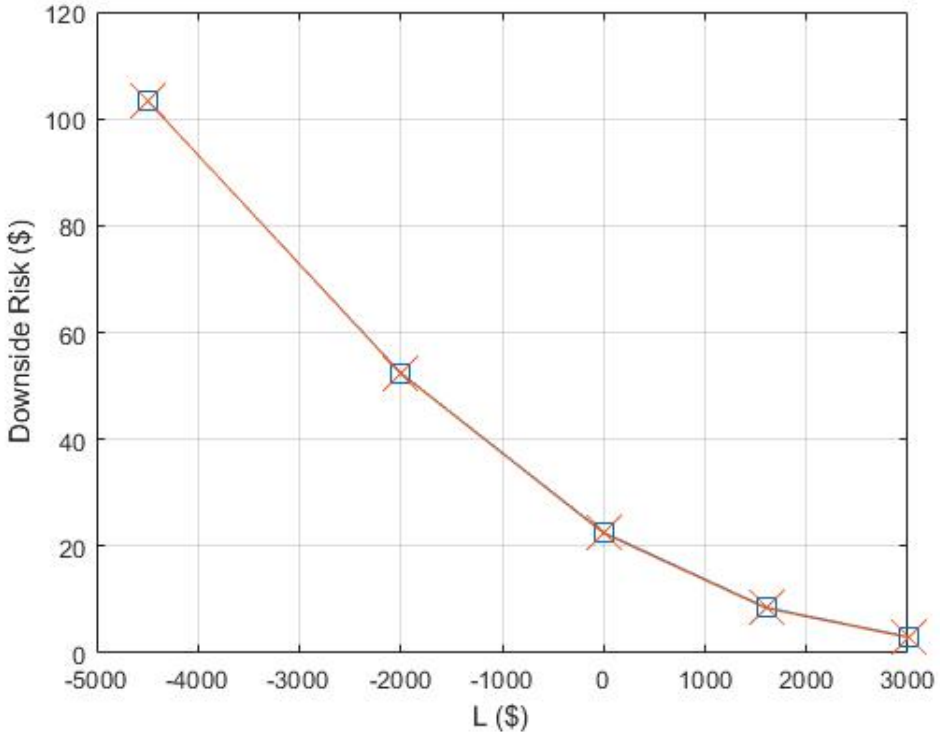

Figure 2: Downside Risk For Portfolio (Initial Value $18,000)

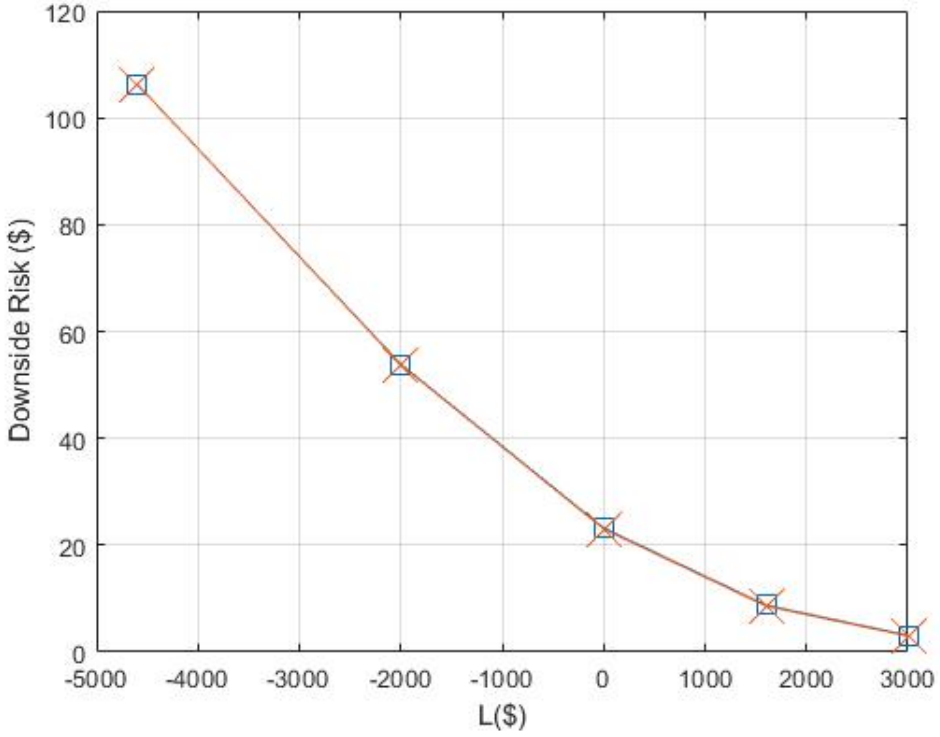

Figure 3: Downside Risk For Portfolio (Initial Value $18,500)

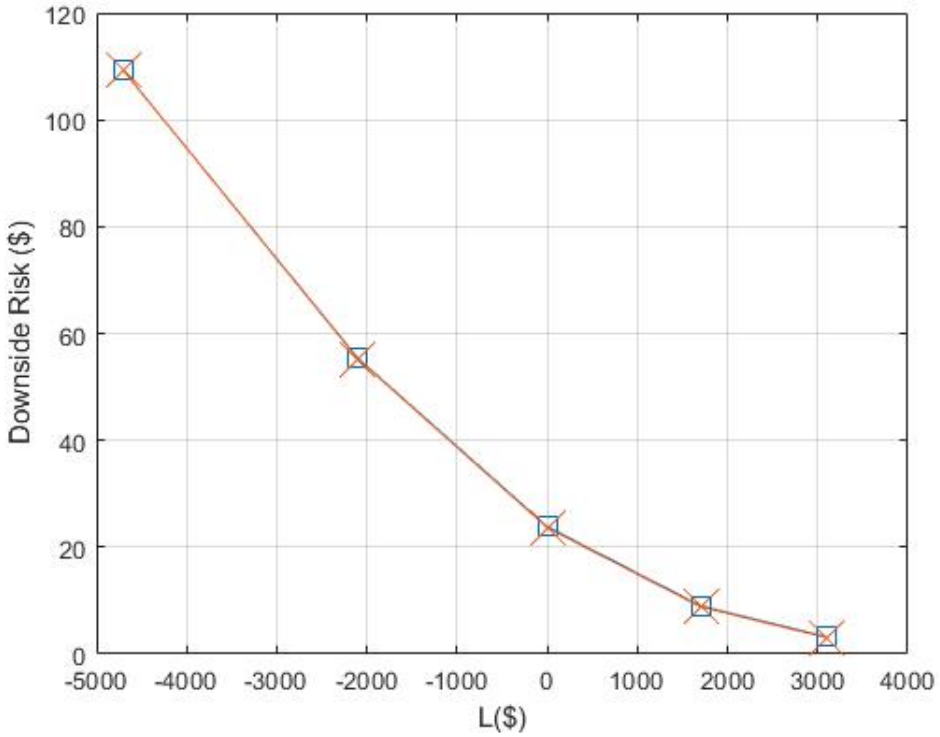

Figure 4: Downside Risk For Portfolio (Initial Value $19,000)

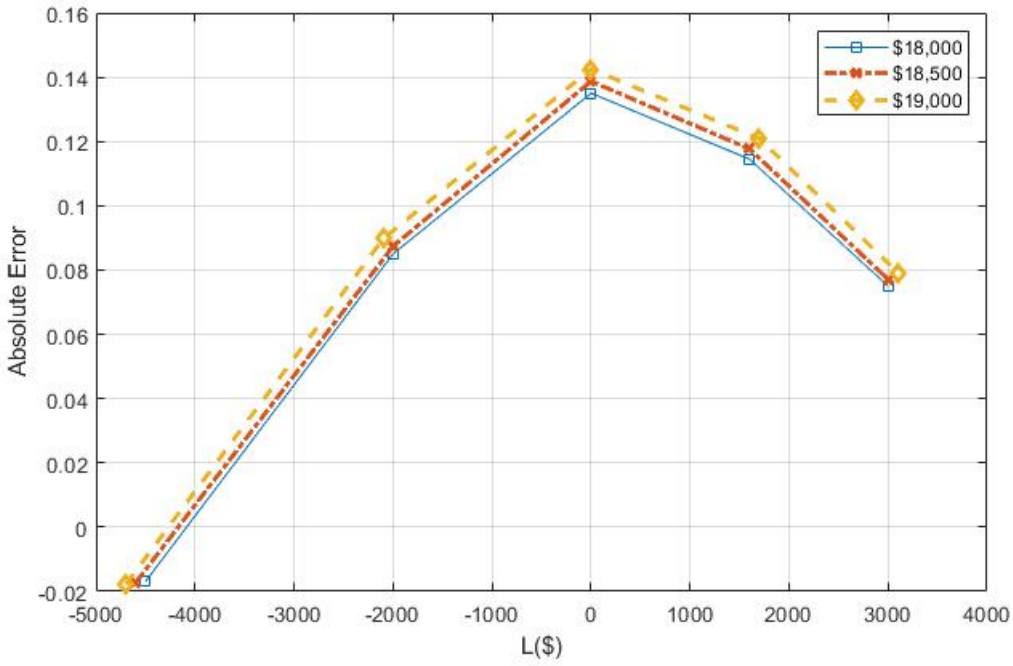

Figure 5: Absolute Error In Downside Risk Measurement For Different L and Initial Portfolio Values $18,000, $18,500, $19,000

The results of the numerical experiments are provided in Tables 1-4 and Figures 1-5, where RS denotes the risk measurement under our regime switching model and MC denotes risk measurement using Monte Carlo simulation. In Figures 2-4 the results associated with the crosses relate to RS, whereas the results associated with the squares are associated with MC. In MC simulation we simulate the stochastic process to obtain the distribution, and then utilise the distribution to calculate the downside risk measure, as well as additional risk measures such as VaR (Value at Risk).

In Figure 1 we provide a set of sample paths for the portfolio (with initial value \$18,000), over the time period of risk measurement (T=0.25). As can be seen from Figure 1, the stochastic process (Heston model) leads to a wide range of values and dynamics over the time period. In particular, we notice that the variation in values increases over time. Hence Figure 1 demonstrates the importance of having risk measures, that is risk measures enable quantification of the probability of different losses in the future.

In Tables 1,3 and 4 we calculate the downside risk measures; for robustness we test across 3 different portfolio values (\$18,000, \$18,500, \$19,000 respectively for each table), and we test across similar as well as different threshold values L across the portfolios. The results of downside risk calculation under RS and MC are given in Tables 1,3 and 4, and plotted in Figures 2-4, respectively. As expected, the downside risk decreases as L increases, since the loss beyond the threshold L should be less likely as L increases. In Table 1 our method shows that our downside risk measures are consistent with the downside risk measures obtained using MC simulation; in fact Figures 2-4 show that the graphs for MC and RS are virtually identical, giving a similar relationship between downside risk and L under both methods. Hence our RS model provides acceptable risk measurement results.

To illustrate that the downside risk measure is a more preferable risk measure (as well as more informative) compared to traditional risk measures, we calculated a number of risk measures in Table 2 for the portfolio's loss distribution. In Table 2 we calculated risk measures for the portfolio with initial value \$18,000, over time period T=0.25 yr, using a number of traditional risk measures (see Szegö (2005) for more information). In particular we calculated variance (which the standard Markowitz risk measure Markowitz (1991)), the Sharpe Ratio Sharpe (1966), n^{th} moments μ_n as well as Value at Risk (VaR) for different quantile values q at the $90\%, 95\%, 98\%, 99\%$ values. As can be seen from Table 2 the Variance is a high figure, implying that the portfolio is high risk. Similarly the Sharpe Ratio (using the current riskless interest rate of 1%) and gives a relatively low value, which implies that the portfolio has a high risk in relation the returns possible. Simillarly the first four moments are calculated and are high figures, suggesting s high risk portfolio. However, the problem with these risk measures is that they do not any indication of the degree of loss involved, consequently they are not

as informative as the downside risk measures. The VaR risk measures are also calculated at different quantiles q, these are more informative than the previous risk measures because they provide an indication of the loss and the associated probabilities. However, the VaR risk measure is not a coherent risk measure (unlike the downside risk measure) and so does not take into account portfolio diversification. Hence VaR can be a misleading measure of risk.

We examine the error in our method compared to the Monte Carlo simulation method by calculating the absolute error. In Tables 1,3 and 4 we provide the absolute error between the Monte Carlo and Regime Switching calculations; these are plotted in Figure 5. The absolute error between Monte Carlo and our method is relatively small, for example for $L = -4500$ in Table 1the absolute error is -0.0169, and so our method is accurate. In Figure 5 we notice that the absolute error is not monotonically increasing or decreasing with L, in fact absolute error tends to decrease with the magnitude of L increases (either for positive or negative values). The initial portfolio value also has a negligible effect on absolute error, with a marginal increase in error as the initial portfolio value decreases. Therefore our method may incur more significant error for downside risk measurement for $|L| \to 0$ and small portfolio values.

8 Conclusion

Regime switching is a valuable modelling method, that is applicable to a range of applications, however, regime switching tends to not model more complex stochastic volatility processes. Consequently, the associated risk measures tend to be unable to provide realistic risk measurement. In this paper, we provide a regime switching model that is able to capture a range of non-trivial stochastic volatility processes. We derive the downside risk measure for this model, with and without jump risk, and so can quantify risk with and without event risk. We also prove that our risk measure can provide consistent risk measurement on the underlying asset model, as the regime switching model converges to this model when the number of regimes tends to infinity. We also provide a discretisation process for the variance process, which is a fundamental risk process in financial models. Moreover, we conduct numerical experiments, with market based parameters, to demonstrate our method and examine the results.

In terms of future work, we would like to investigate other regime switching models and develop models for alternative applications, such as regime switching interest rate models, or regime switching inflation models. Secondly, we would like to investigate other risk measures and their relation to regime switching models, such as spectral risk measures. Finally, we would like to investigate alternative computational methods for regime switching methods, as there is a significant amount of research on computation of Markov processes, hence regime switching research may benefit from applying such research.

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