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CHARACTER EXPANSION OF KAC–MOODY CORRECTION FACTORS

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ABSTRACT. A correction factor naturally arises in the theory of p-adic Kac–Moody groups. In this paper, we expand the correction factor into a sum of irreducible characters of the underlying Kac–Moody algebra. We derive a formula for the coefficients which lie in the ring of power series with integral coefficients. In the case that the Weyl group is a universal Coxeter group, we show that the coefficients are actually polynomials.

1. INTRODUCTION

Let W be a Coxeter group, and consider its Poincaré series

$$\chi(q) := \sum_{w\in W} q^{\ell(w)},$$

where q is an indeterminate and $\ell(w)$ is the length of w. R. Steinberg showed in [St68] that the series $\chi(q)$ represents a rational function in q. When W is the Weyl group of an irreducible, reduced, finite root system Φ, I.G. Macdonald [M72] found the following identity:

$$\sum_{w\in W}\prod_{\alpha\in\Phi^+}\left(\frac{1-qe^{-w\alpha}}{1-e^{-w\alpha}}\right)=\chi(q), \tag{1.1}$$

where Φ^+ is the set of positive roots and e^{β} is a formal exponential associated to β in the root lattice Q. Macdonald's identity reflects the geometry of the flag manifold.

A generalization of the left-hand side of (1.1) to a Kac–Moody root system Φ would be

$$\mathcal{M}(q) := \sum_{w\in W}\prod_{\alpha\in\Phi^+}\left(\frac{1-qe^{-w\alpha}}{1-e^{-w\alpha}}\right)^{m(\alpha)},$$

where $m(\alpha)$ is the multiplicity of α. The identity (1.1) is no longer true for $\mathcal{M}(q)$, and so it is interesting to compute the *correction factor* [1] $\mathcal{M}(q)/\chi(q)$. Macdonald [M03] computed this quotient for the affine Kac–Moody case. The computation turns out to be equivalent to the Macdonald constant term conjecture [M82], which was proven by I. Cherednik in [Ch95].

The correction factor appears in the study of p-adic affine Kac–Moody groups, namely in the formal computation of Fourier coefficients of Eisenstein series and in the study of corresponding Hecke algebras. For example, it was shown by Braverman–Finkelberg–Kazhdan that this correction factor appears in the Gindikin–Karplevich formula for affine Kac–Moody groups [BFK] (see also

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[1] A slight modification of this quotient, denoted by $\mathfrak{m}$, is what Macdonald called the *constant term* in the affine case and is also called the "correction factor" in the literature (see (2.10) for a precise definition).

[BGKP, BKP, GR14, BPGR16]). The correction factor in the general case was studied by Muthiah–Puskas–Whitehead [MPW]. They encoded the data of the correction factor into a collection of polynomials indexed by positive imaginary roots and derived formulas for these polynomials.

In this paper, we study the correction factor $\mathcal{M}(q)/\chi(q)$ for arbitrary Kac–Moody root systems, which we write as a sum of characters $\mathrm{ch}\,(L(\lambda))$ of integrable irreducible representations $L(\lambda)$ of the Kac–Moody algebra $\mathfrak{g}$ with root system Φ. As the first main result of this paper, we prove that the sum is supported on $\lambda \in P^+ \cap Q^-_{\mathrm{im}}$, where P^+ is the set of dominant integral weights and Q^-_{im} is the cone generated by negative imaginary roots. More precisely, we obtain

Theorem 1.1. *Given a Kac–Moody algebra $\mathfrak{g}$, let P^+ denote its set of dominant integral weights and Q^-_{im} its negative imaginary root cone. Then there are $d_\lambda \in \mathbb{Z}[[q]]$, $\lambda \in P^+ \cap Q^-_{\mathrm{im}}$, such that*

$$\mathcal{M}(q)/\chi(q) = \sum_{\lambda \in P^+ \cap Q^-_{\mathrm{im}}} d_\lambda \,\mathrm{ch}\,(L(\lambda)) \,. \tag{1.2}$$

Actually, we prove this result for any W-invariant functions with support in the negative root cone Q^- (see Theorem 2.13). We recover (1.1) as an immediate consequence, since $P^+ \cap Q^-_{\mathrm{im}} = \{0\}$ for finite root systems. This result also explains why the known formulas in the affine case only involve imaginary roots.

The coefficients d_λ are related to the function $H(\mu; q)$, $\mu \in Q$, which was introduced by Kim and Lee [KL11, KL12] in a study of p-adic integrals using canonical/crystal bases from the context of Weyl group multiple Dirichlet series ([Bu12] for a survey). See Definition 3.1 for the definition of $H(\mu; q)$. We prove the following formula (Theorem 3.5):

$$\chi(q)\, d_\lambda = \sum_{w \in W} (-1)^{\ell(w)} H(-w \circ \lambda; q), \tag{1.3}$$

where $w \circ \lambda := w(\lambda + \rho) - \rho$ with a Weyl vector ρ.

Using (1.3), one can compute d_λ explicitly. In particular, in the rank 2 hyperbolic case, we observe that they are actually polynomials in q. Generalizing this observation, we prove that d_λ are always polynomials when W is a universal Coxeter group, or equivalently, when $a_{ij}a_{ji} \geq 4$ for all $i, j \in I$ with the generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ of $\mathfrak{g}$. Formally, we obtain

Theorem 1.2. *Assume that the Weyl group W of $\mathfrak{g}$ is a universal Coxeter group. Then we have $d_\lambda \in \mathbb{Z}[q]$ for all $\lambda \in P^+ \cap Q^-_{\mathrm{im}}$.*

It would be very interesting to see if d_λ are polynomials for arbitrary Kac–Moody root systems. We expect that these coefficients carry important combinatorial, representation-theoretic information, which is yet to be revealed. We hope that we can investigate these issues in the near future.

The main text proceeds as follows. In Section 2 we review standard background material and construct a large ring containing $\mathcal{M}(q)$ equipped with a W-action. We conclude with the statement that W-invariant elements admit a character expansion, which applies in particular to $\mathcal{M}(q)$. In Section 3 we compute the character coefficients in terms of the function H. Though the formula deduced involves an infinite sum, it exhibits a large amount of cancellation and in Section 4 we show that it is in fact a polynomial when W is a universal Coxeter group. In the Appendix, we give compute the coefficients for certain small imaginary roots of a rank 2 hyperbolic Kac–Moody algebra.

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2. Existence of character coefficients

We will use the conventions and terminology of [Ka90]. Let $I = \{1, \cdots, n\}$ and let A be a generalized Cartan matrix with realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$. In particular, the elements of the set $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ (resp. $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$) are the simple roots (resp. simple coroots). The root lattice Q (resp. positive root cone Q^+) is the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\geq 0}$-span) of Π. We set $Q^- = -Q^+$. A partial order $\geq$ on $\mathfrak{h}^*$ is defined by $\mu \geq \nu$ if $\mu - \nu \in Q^+$. We say $\alpha \in \mathfrak{h}^*$ is positive (resp. negative) if $\alpha > 0$ (resp. $\alpha < 0$).

Let $\mathfrak{g}$ be the Kac–Moody algebra associated to A, which admits the root space decomposition $\mathfrak{g} = \oplus_{\alpha \in Q} \mathfrak{g}_\alpha$, with $\mathfrak{g}_0 = \mathfrak{h}$. Given $\alpha \in Q$, its multiplicity $m(\alpha)$ is the dimension of the vector space $\mathfrak{g}_\alpha$. A non-zero $\alpha \in Q$ is a root if $m(\alpha) \neq 0$. We will denote the set of roots by Φ, and the set of positive (resp. negative) roots by Φ^+ (resp. Φ^-).

Let W denote the Weyl group of $\mathfrak{g}$, which is the subgroup of $\mathrm{Aut}\,(\mathfrak{h}^*)$ generated by the simple reflections s_i, $i \in I$. A root $\alpha \in \Phi$ is called real if there is $w \in W$ such that $w\alpha$ is a simple root. A root that is not real is called imaginary. If α is real, then $m(\alpha) = 1$. The set of real (resp. imaginary) roots is denoted by Φ_{re} (resp. Φ_{im}), and the set of positive real (resp. positive imaginary) roots is denoted by Φ^+_{re} (resp. Φ^+_{im}).

Let q denote a formal variable, and let $\mathbb{Z}[[q]]$ be the ring of power series in q with integer coefficients. Recall that $f(q) \in \mathbb{Z}[[q]]$ is invertible if and only if the constant term $f(0)$ of f is equal to ± 1. The inverse of a unit in $\mathbb{Z}[[q]]$ will be written as a fraction whenever it is convenient. For example, we write

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots .$$

Example 2.1. The Poincaré series of the Weyl group W is defined as follows:

$$\chi(q) = \sum_{w \in W} q^{\ell(w)} \in \mathbb{Z}[[q]], \tag{2.1}$$

where the length $\ell(w)$ of $w \in W$ is the minimal ℓ such that $w = s_{i_1} \cdots s_{i_\ell}$ is a product of simple reflections. As the only word of length 0 is the identity element, the constant term of $\chi(q)$ is 1. Thus, $\chi(q) \in \mathbb{Z}[[q]]^\times$.

Notation. To each $\lambda \in \mathfrak{h}^*$, we associate a formal exponential denoted by e^λ, and define $e^\lambda e^\mu = e^{\lambda+\mu}$ for $\lambda, \mu \in \mathfrak{h}^*$. Let $\mathbb{Z}((q))$ denote the ring of Laurent series with integral coefficients, and let R be a subring of $\mathbb{Z}((q))$. We denote by $\mathcal{S}(R)$ the additive group of formal sums $\sum_{\lambda \in \mathfrak{h}^*} a_\lambda e^\lambda$ with $a_\lambda \in R$ for all $\lambda \in \mathfrak{h}^*$.

Definition 2.2. The *support* of a formal sum $\sum_{\lambda \in \mathfrak{h}^*} a_\lambda e^\lambda \in \mathcal{S}(R)$ is the set of $\lambda \in \mathfrak{h}^*$ such that $a_\lambda \neq 0$.

If $f = \sum_{\lambda \in Q} a_\lambda e^\lambda$ is a unit of $\mathcal{S}(R)$ and has support in a translate of Q^-, then f has a unique product expansion as in [MPW, Proposition 2.2]:

$$\sum_{\lambda \in Q} a_\lambda e^\lambda = u e^{\lambda_0} \prod_{\lambda \in Q^- \setminus \{0\}} \prod_{n \in \mathbb{Z}} (1 - q^n e^\lambda)^{m(\lambda, n)}, \tag{2.2}$$

for some $u \in R^\times$, $\lambda_0 \in Q$ and $m(\lambda, n) \in \mathbb{Z}$ such that, for every λ, the set $\{n \in \mathbb{Z} : m(\lambda, n) \neq 0\}$ is bounded below.

Definition 2.3 (Section 2.3 in [MPW])**.** A product of the form (2.2) is called a *good product with coefficients in* R if all λ appearing in its factors are multiples of roots $\alpha \in \Phi$, and the set of factors corresponding to any real root $\alpha \in \Phi_{\text{re}}$ is finite. We will denote by $\mathcal{G}(R)$ the multiplicative group of good products with coefficients in R.

An element of $\mathcal{G}(R)$ expands to a formal sum in $\mathcal{S}(R)$ by definition. The notion of a good product is introduced, in part, to define the action of W as below.

Definition 2.4. We define an action of W on $\mathcal{G}(R)$ by extending the following action on the factors of (2.2) multiplicatively:

$$w(1-q^n e^{\lambda}) = \begin{cases} 1-q^n e^{w(\lambda)}, & w(\lambda)<0, \\ (-q^n e^{w(\lambda)})(1-q^{-n}e^{-w(\lambda)}), & w(\lambda)>0, \end{cases} \tag{2.3}$$

for $w \in W$. Given $f \in \mathcal{G}(R)$, we will sometimes write $f^w = w(f)$. We will denote by $\mathcal{G}^W(R)$ the ring of W-invariant elements of $\mathcal{G}(R)$.

Define the negative imaginary cone Q^-_{im} to be the cone generated by negative imaginary roots. Then we have $Q^-_{\text{im}} = \bigcap_{w\in W} w(Q^-)$. Thus if $f \in \mathcal{G}^W(R)$ is supported on Q^-, then it is in fact supported on Q^-_{im}. It was noted in [MPW] that, for $w \in W$ and $f = \sum_{\lambda\in\mathfrak{h}^*} a_\lambda e^{\lambda} \in \mathcal{G}(R)$, we have

$$w(f) = \sum_{\lambda\in\mathfrak{h}^*} a_\lambda e^{w\lambda}. \tag{2.4}$$

Remark 2.5. The set of $f \in \mathcal{S}(R)$ supported on Q^- is not closed under the action of W defined by (2.4), but $\mathcal{G}(R)$ is.

The basic good product in this paper is

$$\Delta := \prod_{\alpha\in\Phi^+} \left(\frac{1-qe^{-\alpha}}{1-e^{-\alpha}} \right)^{m(\alpha)}. \tag{2.5}$$

Here $\frac{1-qe^{-\alpha}}{1-e^{-\alpha}} = 1 + \sum_{n\geq 1}(1-q)e^{-n\alpha}$, and it is clear that $\Delta \in \mathcal{G}(\mathbb{Z}[q])$.

Since $m(\alpha)=1$ for $\alpha \in \Phi^+_{\text{re}}$, we set

$$\Delta_{\text{re}} := \prod_{\alpha\in\Phi^+_{\text{re}}} \left(\frac{1-qe^{-\alpha}}{1-e^{-\alpha}} \right), \qquad \Delta_{\text{im}} := \prod_{\alpha\in\Phi^+_{\text{im}}} \left(\frac{1-qe^{-\alpha}}{1-e^{-\alpha}} \right)^{m(\alpha)} \tag{2.6}$$

so that we have

$$\Delta = \Delta_{\text{re}}\Delta_{\text{im}}.$$

Finally, define

$$\boxed{\mathcal{M}(q) := \sum_{w\in W} \Delta^w.} \tag{2.7}$$

Clearly, $\mathcal{M}(q)$ is W-invariant since it is the sum of W-action on Δ.

Lemma 2.6. *The formal sum $\mathcal{M}(q)$ is a W-invariant good product with coefficients in $\mathbb{Z}[[q]]$, i.e. $\mathcal{M}(q) \in \mathcal{G}^W(\mathbb{Z}[[q]])$. Moreover, $\mathcal{M}(q)$ is supported on Q^-_{im} and has the constant term equal to $\chi(q)$.*

Proof. Since the set Φ^+_{im} is W-invariant, and $m(w\alpha) = m(\alpha)$ for $w \in W$ and $\alpha \in \Phi$, we have $\Delta^w_{\text{im}} = \Delta_{\text{im}}$. It follows that $\Delta^w = (\Delta_{\text{re}}\Delta_{\text{im}})^w = \Delta^w_{\text{re}}\Delta_{\text{im}}$. By Definition 2.4, we have

$$w\left(\frac{1-qe^{-\alpha}}{1-e^{-\alpha}}\right) = \begin{cases} \dfrac{1-qe^{-w(\alpha)}}{1-e^{-w(\alpha)}} & \text{if } w(\alpha) > 0, \\[2ex] \dfrac{qe^{-w(\alpha)}(1-q^{-1}e^{w(\alpha)})}{e^{-w(\alpha)}(1-e^{w(\alpha)})} = \dfrac{q(1-q^{-1}e^{w(\alpha)})}{1-e^{w(\alpha)}} & \text{if } w(\alpha) < 0, \end{cases}$$

for $w \in W$ and $\alpha \in \Phi^+_{\text{re}}$. One can immediately see that the sum $\mathcal{M}(q)$ is supported on Q^-. Since

$$\frac{1-qe^{-\alpha}}{1-e^{-\alpha}} = 1 + \sum_{n\geq 1}(1-q)e^{-n\alpha} \quad \text{and} \quad \frac{q(1-q^{-1}e^{-\alpha})}{1-e^{-\alpha}} = \frac{q-e^{-\alpha}}{1-e^{-\alpha}} = q - \sum_{n\geq 1}(1-q)e^{-n\alpha}, \tag{2.8}$$

we see that Δ^w is a good product with coefficients in $\mathbb{Z}[q]$, i.e. $\Delta^w \in \mathcal{G}(\mathbb{Z}[q])$.

Now we check that the coefficient of $e^{-\beta}$ in $\mathcal{M}(q) = \sum_{w\in W}\Delta^w$ is an element of $\mathbb{Z}[[q]]$ for $\beta \in Q^+$. For $w \in W$, define

$$\Phi(w) := \{\alpha \in \Phi^+_{\text{re}} \mid w(\alpha) < 0\} = \Phi^+ \cap w^{-1}\Phi^-.$$

It is well-known that $|\Phi(w)| = \ell(w)$. Thus we have

$$\begin{aligned} \Delta^w_{\text{re}} &= \prod_{\alpha\in\Phi(w^{-1})}\left(\frac{q-e^{-\alpha}}{1-e^{-\alpha}}\right)\prod_{\alpha\in\Phi^+_{\text{re}}\setminus\Phi(w^{-1})}\left(\frac{1-qe^{-\alpha}}{1-e^{-\alpha}}\right) \\ &= q^{\ell(w)}\prod_{\alpha\in\Phi(w^{-1})}\left(\frac{1-q^{-1}e^{-\alpha}}{1-e^{-\alpha}}\right)\prod_{\alpha\in\Phi^+_{\text{re}}\setminus\Phi(w^{-1})}\left(\frac{1-qe^{-\alpha}}{1-e^{-\alpha}}\right). \end{aligned} \tag{2.9}$$

For $\beta \in Q^+$, the coefficient of $e^{-\beta}$ in $\Delta^w = \Delta^w_{\text{re}}\Delta_{\text{im}}$, a priori an element in $\mathbb{Z}[[q]]$, is of the form

$$q^{\ell(w)}p_{\beta,w}$$

for some $p_{\beta,w} \in \mathbb{Z}((q))$. Recall the *height* of $\beta = \sum_{i=1}^n m_i\alpha_i \in Q^+$, $m_i \geq 0$, is defined to be

$$\text{ht}(\beta) := \sum_{i=1}^n m_i.$$

It is easy to observe from (2.8) and (2.9) the crude estimate that the degrees of $p_{\beta,w}$ in q^{-1} and q are both bounded by $\text{ht}(\beta)$. Thus we have $p_{\beta,w} \in \mathbb{Z}[q, q^{-1}]$. Moreover q^m appears in $q^{\ell(w)}p_{\beta,w}$ only if $\ell(w) \leq m + \text{ht}(\beta)$. Since there are only finitely many $w \in W$ of a given length, we see that

$$\mathcal{M}(q) = \sum_{w\in W}\Delta^w = \sum_{\beta\in Q^+}\left(\sum_{w\in W}q^{\ell(w)}p_{\beta,w}\right)e^{-\beta}$$

with the coefficient of $e^{-\beta}$ given by a well-defined sum

$$\sum_{w\in W}q^{\ell(w)}p_{\beta,w} \in \mathbb{Z}[[q]].$$

In particular, when $\beta = 0$, we have $p_{0,w} = 1$ for all $w \in W$ and the constant term of $\mathcal{M}(q)$ is equal to $\sum_{w\in W}q^{\ell(w)} = \chi(q)$.

We have already seen that $\mathcal{M}(q)$ is supported on Q^- at the beginning of the proof. Since $\mathcal{M}(q)$ is also W-invariant, it is supported on Q^-_{im}. (See the paragraph after Definition 2.4.)

Using [MPW, Proposition 2.2], we may write $\mathcal{M}(q)$ as a product of the form (2.2) with $\lambda_0 = 0$. Since $\mathcal{M}(q)$ is supported on Q^-_{im}, no factor corresponding to a real root arises in the product and hence $\mathcal{M}(q)$ is a good product. □

Remark 2.7. (1) We have the following identity in $\mathcal{G}^W(\mathbb{Z}[[q]])$:

$$\mathfrak{m}\mathcal{M}(q) = \Delta_{\mathrm{im}}\chi(q), \tag{2.10}$$

where $\mathfrak{m}$ is as defined in [MPW, equation (3.5)]. Each of $\mathfrak{m}^{-1}$, Δ_{im} and $\mathcal{M}(q)$ expands to a formal sum supported on Q^-_{im}.

(2) In the paper [BPGR19], it was pointed out that $\mathcal{M}(q)$ is not an element of $\mathcal{G}^W(\mathbb{Z}[q, q^{-1}])$ but an element of $\mathcal{G}^W(\mathbb{Z}((q)))$. As a refinement, Lemma 2.6 shows that $\mathcal{M}(q) \in \mathcal{G}^W(\mathbb{Z}[[q]])$.

Now we move on to study a character expansion of an element in $\mathcal{G}^W(\mathbb{Z}[[q]])$.

Definition 2.8. Fix a Weyl vector $\rho \in \mathfrak{h}^*$, i.e. a vector satisfying $\rho(\alpha_i^\vee) = 1$, for all $i \in I$. The circle action[2] of W on $\mathfrak{h}^*$ is defined by

$$w \circ \lambda = w(\lambda + \rho) - \rho. \tag{2.11}$$

Example 2.9. We have

$$w \circ 0 = w\rho - \rho, \tag{2.12}$$

which can be written as a sum of negative roots. Indeed, one has

$$\rho - w\rho = \sum_{\alpha \in \Phi(w^{-1})} \alpha, \tag{2.13}$$

where, for $w \in W$,

$$\Phi(w) := \Phi^+ \cap w^{-1}\Phi^-. \tag{2.14}$$

Denote by P the weight lattice of $\mathfrak{g}$, and by $P^+ \subset P$ the subset of dominant integral weights. For $\lambda \in P$, define

$$\pi^\lambda := \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w\rho}}.$$

Recall the denominator identity

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w\rho - \rho} = \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{m(\alpha)}. \tag{2.15}$$

For $\lambda \in P$, define

$$\xi^\lambda := \sum_{w \in W} (-1)^{\ell(w)} e^{w \circ \lambda}. \tag{2.16}$$

Lemma 2.10.

(1) For $w \in W$, we have

$$w\left(\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{m(\alpha)}\right) = (-1)^{\ell(w)} e^{\rho - w\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{m(\alpha)}.$$

[2] This action is slightly different to the action with the same notation in [KL12].

(2) For $\lambda \in P$ and $w \in W$, we have

$$\xi^\lambda = (-1)^{\ell(w)} \xi^{w \circ \lambda},$$
$$\pi^\lambda = (-1)^{\ell(w)} \pi^{w \circ \lambda}.$$

Proof. (1) From the denominator identity (2.15), we have

$$\begin{aligned} w\left(\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{m(\alpha)}\right) &= \sum_{w_1\in W}(-1)^{\ell(w_1)}e^{ww_1\rho-w\rho} \\ &= \sum_{w_1\in W}(-1)^{\ell(w)+\ell(ww_1)}e^{ww_1\rho-\rho}e^{\rho-w\rho} \\ &= (-1)^{\ell(w)}e^{\rho-w\rho}\sum_{w_1\in W}(-1)^{\ell(ww_1)}e^{ww_1\rho-\rho} \\ &= (-1)^{\ell(w)}e^{\rho-w\rho}\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{m(\alpha)}. \end{aligned}$$

(2) Let $w \circ \lambda = \mu$. Then $w(\lambda+\rho) = \mu+\rho$. Now we have

$$\begin{aligned} \sum_{w_1\in W}(-1)^{\ell(w_1)}e^{w_1(\lambda+\rho)} &= \sum_{w_1\in W}(-1)^{\ell(w)+\ell(w_1w^{-1})}e^{w_1w^{-1}w(\lambda+\rho)} \\ &= (-1)^{\ell(w)}\sum_{w_1\in W}(-1)^{\ell(w_1w^{-1})}e^{w_1w^{-1}(\mu+\rho)}. \end{aligned}$$

Multiplying both sides by $e^{-\rho}$, we get $\xi^\lambda = (-1)^{\ell(w)}\xi^\mu = (-1)^{\ell(w)}\xi^{w\circ\lambda}$. Dividing both sides by $\sum_{w_1\in W}(-1)^{\ell(w_1)}e^{w_1\rho-\rho}$, we obtain $\pi^\lambda = (-1)^{\ell(w)}\pi^{w\circ\lambda}$. □

Consider the following subset of Q^-:

$$Q' := \bigcap_{w\in W} w \circ Q^-. \tag{2.17}$$

The Weyl group W acts on Q' by the circle action, and so $Q^-_{\mathrm{im}} \subset Q'$.

Lemma 2.11. *Assume that $\lambda \in Q'$. Then the following hold.*

1. *There exists a unique $\mu \in Q^-$ and $v \in W$ such that $\mu+\rho \in P^+$ and $v \circ \lambda = \mu$.*
2. *The stabilizer subgroup*
$$W^\circ_\lambda := \{w \in W : w \circ \lambda = \lambda\}$$
is generated by reflections in W.
3. *If $\lambda \in P^+ \cap Q^-$, then $W^\circ_\lambda = \{1\}$.*

Proof. Write $\lambda = \sum_i m_i\alpha_i$ with $m_i \le 0$ for all i. If $\lambda+\rho \in P^+$, there is nothing to prove. If not, there exists j such that $\lambda(\alpha_j^\vee) \le -2$. We have

$$s_j \circ \lambda = s_j(\lambda+\rho) - \rho = \lambda - (\lambda(\alpha_j^\vee)+1)\alpha_j \in Q^-.$$

Since $\lambda(\alpha_j^\vee)+1 < 0$, we have $m_j < m_j - (\lambda(\alpha_j^\vee)+1) \le 0$. If $(s_j \circ \lambda)+\rho$ is in P^+, we are done. Otherwise, repeat the process with replacing λ with $s_j \circ \lambda$. Since the coefficients are increasing and bounded above by 0, this process must end.

Assume that $\lambda+\rho \in P^+$. Suppose that $w \circ \lambda = \mu$ and $\mu+\rho \in P^+$ for $w = s_{i_1}s_{i_2}\cdots s_{i_\ell} \neq 1$, a reduced expression. Then we have $w(\lambda+\rho) = \mu+\rho$. Since $(\lambda+\rho)(\alpha_{i_\ell}^\vee) \geq 0$, we have $(\mu+\rho)(w(\alpha_{i_\ell}^\vee)) \geq 0$. Since $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ is a reduced expression, we get $w(\alpha_{i_\ell}^\vee) < 0$, and $(\mu+\rho)(w(\alpha_{i_\ell}^\vee)) \leq 0$. Thus $(\mu+\rho)(w(\alpha_{i_\ell}^\vee)) = 0$ and $(\lambda+\rho)(\alpha_{i_\ell}^\vee) = 0$. Hence $s_{i_\ell}(\lambda+\rho) = \lambda+\rho$ and $s_{i_\ell} \circ \lambda = \lambda$. By induction, we obtain $\mu = \lambda$, which completes a proof of (1). We have also shown that the subgroup W_λ° is generated by simple reflections for $\lambda+\rho \in P^+$.

Assume that $\lambda \in P^+ \cap Q^-$, and suppose that $w \circ \lambda = \lambda$ for $w = s_{i_1}s_{i_2}\cdots s_{i_\ell} \neq 1$, a reduced expression. Then $(\lambda+\rho)(\alpha_{i_\ell}^\vee) > 0$ and the above argument leads to a contradiction. Thus we must have $w = 1$. This proves (3).

Now assume that $\lambda \in Q'$. By Lemma 2.11(1), there exists $v \in W$ such that $v \circ \lambda + \rho \in P^+$. Then $W_{v\circ\lambda}^\circ$ is generated by simple reflections s_i. Hence W_λ° is generated by $v^{-1}s_i v$, which are reflections. This completes a proof of (2). □

Lemma 2.12. *Assume that $\lambda \in Q'$. The series $\xi^\lambda \in \mathcal{G}(\mathbb{Z})$ defined in (2.16) is non-zero if and only if the stabilizer subgroup W_λ° of λ under the circle action is trivial.*

Proof. Suppose that $\xi^\lambda = 0$. Then the term e^λ cancels with $(-1)^{\ell(w)}e^{w\circ\lambda}$ for some $w \neq 1$. In particular, $\lambda = w \circ \lambda$, and the stabilizer subgroup W_λ° is not trivial.

Conversely, assume that the stabilizer subgroup W_λ° is not trivial. By Lemma 2.11 there exists a reflection $s \in W_\lambda^\circ$ such that $s \circ \lambda = \lambda$. It follows from Lemma 2.10 that $\xi^\lambda = (-1)^{\ell(s)}\xi^{s\circ\lambda} = -\xi^\lambda$. Hence $\xi_\lambda = 0$. □

Given $\lambda \in P^+$, let $L(\lambda)$ denote the irreducible highest weight module of $\mathfrak{g}$ with highest weight λ. The module $L(\lambda)$ admits a weight space decomposition $L(\lambda) = \oplus_{\mu\in\mathfrak{h}^*} L_\mu$. The character $\mathrm{ch}(L(\lambda))$ of $L(\lambda)$ is defined by

$$\mathrm{ch}(L(\lambda)) = \sum_{\mu\in\mathfrak{h}^*} (\dim L_\mu)\, e^\mu. \tag{2.18}$$

If $\lambda \in P^+$, then by [Ka90] we have

$$\pi^\lambda = \mathrm{ch}\,(L(\lambda))\,. \tag{2.19}$$

Theorem 1.1 is a consequence of the following result.

Theorem 2.13. *Given a Kac–Moody algebra $\mathfrak{g}$, let P^+ denote its set of dominant integral weights and Q^-_{im} its negative imaginary root cone. If $f \in \mathcal{G}^W(\mathbb{Z}[[q]])$ is such that $\mathrm{supp}(f) \subset Q^-$, then there are $c_\lambda \in \mathbb{Z}[[q]]$, $\lambda \in P^+ \cap Q^-_{\mathrm{im}}$, such that*

$$f = \sum_{\lambda\in P^+\cap Q^-_{\mathrm{im}}} c_\lambda\, \mathrm{ch}\,(L(\lambda))\,. \tag{2.20}$$

Proof. Since f is supported on Q^-, we may write the following product as a sum supported on Q^-:

$$\Xi = f \cdot \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})^{m(\alpha)} = \sum_{\beta\in Q^-} c_\beta e^\beta. \tag{2.21}$$

As f is invariant under W, it follows from Lemma 2.10(1) that

$$w\left(f\cdot\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{m(\alpha)}\right)=\sum_{\beta\in Q^-}c_\beta e^{w\beta} \tag{2.22}$$
$$=(-1)^{\ell(w)}e^{\rho-w\rho}f\cdot\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{m(\alpha)}=\sum_{\gamma\in Q^-}(-1)^{\ell(w)}c_\gamma e^{\rho-w\rho+\gamma}.$$

Comparing coefficients, we see that for $\beta\in Q^-$,

$$c_\beta=(-1)^{\ell(w)}c_{w\circ\beta}.$$

Moreover, $c_\beta=0$ unless $\beta\in Q'$, i.e. Ξ is supported on Q'. If $\lambda+\rho\in P^+$ and $\lambda\notin P^+\cap Q^-$ for $\lambda\in Q^-$, then there exists $\alpha_i^\vee$ such that $(\lambda+\rho)(\alpha_i^\vee)=0$ and $s_i\circ\lambda=\lambda$. Thus $\xi^\lambda=0$ by Lemma 2.12.

By Lemma 2.11(1) and the above argument, we group the terms of equation (2.21) to get a sum over $P^+\cap Q^-$, which is the subset of representatives λ of the $\circ$-action of W on Q' such that $\xi^\lambda\neq 0$:

$$\Xi=\sum_{\beta\in Q'}c_\beta e^\beta=\sum_{\lambda\in P^+\cap Q^-}c_\lambda\xi^\lambda. \tag{2.23}$$

On the other hand, for $\lambda\in P^+$, Weyl's character formula implies

$$\xi^\lambda=\mathrm{ch}\,(L(\lambda))\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{m(\alpha)}. \tag{2.24}$$

The result follows from combining (2.23) with (2.24), noting that f is in fact supported on Q^-_{im}. □

Remark 2.14. As mentioned in the introduction, we recover (1.1) as an immediate consequence of Theorem 2.13, since $P^+\cap Q^-_{\mathrm{im}}=\{0\}$ for finite root systems. In the affine case, we have $P^+\cap Q^-_{\mathrm{im}}=\mathbb{Z}_{\leq 0}\cdot\delta$ with the minimal positive imaginary root δ, and the theorem shows that the right-hand side of (2.20) only involves imaginary roots.

3. A formula for the character coefficients

In this section, we derive a formula for the coefficients in the expansion of $\mathcal{M}(q)$ into a sum of characters. We begin with the definition of a function which will play an important role in what follows.

Definition 3.1 ([KL11, KL12])**.** The function $H:Q^+\to\mathbb{Z}[q]$ is defined by the generating series in $\mathcal{G}(\mathbb{Z}[q])$:

$$\sum_{\mu\in Q^+}H(\mu;q)e^{-\mu}=\prod_{\alpha\in\Phi^+}(1-qe^{-\alpha})^{m(\alpha)}, \tag{3.1}$$

where $m(\alpha)$ is the multiplicity of α. When we do not need to specify q, we will frequently write $H(\mu)=H(\mu;q)$.

Remark 3.2. In [KL11, KL12], the function H was denoted by H_ρ. See (2-13) in [KL12].

Definition 3.3. Let $\mu\in Q^+$, and $\mathscr{P}:=\{(\alpha;i):\alpha\in\Phi^+,i=1,2,\dots,m(\alpha)\}$. An *admissible partition* of μ is a finite set $\mathfrak{p}\subset\mathscr{P}$ such that $\sum_{(\alpha,i)\in\mathfrak{p}}\alpha=\mu$. Let $\mathcal{P}(\mu)$ be the set of admissible partitions of μ. Given $\mathfrak{p}\in\mathcal{P}(\mu)$, we will refer to an element $(\alpha,i)\in\mathfrak{p}$ as *part* of $\mathfrak{p}$, and denote the number of parts in $\mathfrak{p}$ by $|\mathfrak{p}|$.

Examples of admissible partitions are given in Appendix A.

Lemma 3.4. *We have*

$$H(\mu) = \sum_{\mathfrak{p}\in\mathcal{P}(\mu)} (-q)^{|\mathfrak{p}|}. \tag{3.2}$$

Proof. Equation (3.2) follows from expanding the product in equation (3.1) and computing the coefficient of $e^{-\mu}$. □

We now prove equation (1.3), which we state below as a theorem for ease of reference.

Theorem 3.5. *For $\lambda \in P^+ \cap Q^-_{\mathrm{im}}$, define d_λ by equation (1.2). Then we have*

$$\chi(q)\, d_\lambda = \sum_{w\in W} (-1)^{\ell(w)} H(-w\circ\lambda). \tag{3.3}$$

Proof. By definition, we have

$$\begin{aligned}\mathcal{M}(q) = \sum_{w\in W} \Delta^w &= \sum_{w\in W} \prod_{\alpha\in\Phi^+} \frac{(1-qe^{-w\alpha})^{m(\alpha)}}{(1-e^{-w\alpha})^{m(\alpha)}} \\ &= \sum_{w\in W} \frac{\sum_{\mu\in Q^+} H(\mu)e^{-w\mu}}{\prod_{\alpha\in\Phi^+}(1-e^{-w\alpha})^{m(\alpha)}} \\ &= \sum_{\mu\in Q^-} H(-\mu) \sum_{w\in W} \frac{e^{w\mu}}{\prod_{\alpha\in\Phi^+}(1-e^{-w\alpha})^{m(\alpha)}}.\end{aligned}$$

Using Lemma 2.10 (1), we deduce that

$$\mathcal{M}(q) = \frac{1}{\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})^{m(\alpha)}} \sum_{\mu\in Q^-}\sum_{w\in W} (-1)^{\ell(w)} H(-\mu) e^{w\circ\mu}.$$

As in the proof of Theorem 2.13, put

$$\Xi := \mathcal{M}(q) \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})^{m(\alpha)} = \sum_{\mu\in Q^-}\sum_{w\in W} (-1)^{\ell(w)} H(-\mu) e^{w\circ\mu}.$$

Since Ξ is supported on Q', we may rewrite the above double sum as

$$\Xi = \sum_{\beta\in Q'}\sum_{w\in W} (-1)^{\ell(w)} H(-w\circ\beta) e^{\beta}.$$

The theorem then follows from (2.23). □

Example 3.6. Given $w \in W$, write $w = s_{i_1} \cdots s_{i_\ell}$ as a reduced expression. If $\Phi(w^{-1})$ is as defined in equation (2.14), then

$$\Phi(w^{-1}) = \left\{ \alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1}\cdots s_{i_{\ell-1}}(\alpha_{i_\ell}) \right\}, \tag{3.4}$$

and

$$w\circ 0 = \rho - w\rho = \sum_{\alpha\in\Phi(w^{-1})} \alpha = \alpha_{i_1} + s_{i_1}(\alpha_{i_2}) + \cdots + s_{i_1}\cdots s_{i_{\ell-1}}(\alpha_{i_\ell}). \tag{3.5}$$

Suppose that

$$\rho - w\rho = \beta_1 + \beta_2 + \cdots + \beta_k$$

for some positive roots $\beta_1, \ldots, \beta_k \in \Phi^+$. Note that we have

$$s_{i_1}(\rho - w\rho) \notin Q^+.$$

Since s_{i_1} keeps Φ^+ except α_{i_1}, one of the β_i's must be equal to α_{i_1}. Then $s_{i_1}(\rho - w\rho - \alpha_{i_1})$ is equal to $\rho - w'\rho$ where $w' = s_{i_2} \cdots s_{i_\ell}$. Arguing by induction on $\ell(w)$, we deduce that (3.5) is the unique decomposition of $w \circ 0$ into a sum of positive roots. Now it follows from (3.2) that

$$H(-w \circ 0) = H(\rho - w\rho) = (-q)^{\ell(w)}, \tag{3.6}$$

and so the formula (3.3) yields

$$d_0 = 1.$$

Lemma 3.7. *For all nonzero $\lambda \in P^+ \cap Q^-$ and $w \in W$, the coefficients of $H(-w \circ \lambda)$ sum to zero.*

Proof. From [KL12, Lemma 3.18], we have

$$H(\mu; 1) = \begin{cases} (-1)^{\ell(w)}, & \text{if } \rho - w\rho = \mu \text{ for some } w \in W, \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Therefore, it suffices to show that

$$-w \circ \lambda = -(w(\lambda + \rho) - \rho) = \rho - w(\lambda + \rho) \neq \rho - v\rho \tag{3.8}$$

for any $v \in W$. Equation (3.8) is equivalent to $\lambda + \rho \neq w^{-1} v\rho$, and so it is enough to show, for any $v \in W$,

$$\lambda \neq v\rho - \rho.$$

If $v = 1$ there is nothing to prove. Consider an arbitrary $v \neq 1$, and write v^{-1} as a reduced word $s_{i_1} \cdots s_{i_k}$. Then we have

$$\rho(v^{-1}\alpha_{i_k}^\vee) < 0,$$

and

$$(v\rho - \rho)(\alpha_{i_k}^\vee) = v\rho(\alpha_{i_k}^\vee) - \rho(\alpha_{i_k}^\vee) = \rho(v^{-1}\alpha_{i_k}^\vee) - 1 \leq -2.$$

Thus $v\rho - \rho \notin P^+$. Since $\lambda \in P^+$, we have $\lambda \neq v\rho - \rho$. ☐

Definition 3.8. Let $\lambda \in Q^-$ and

$$\mathfrak{p} = \{(\beta_1; m_1), (\beta_2; m_2), \ldots, (\beta_t; m_t)\} \in \mathcal{P}(-\lambda). \tag{3.9}$$

Given $w \in W$, we define

$$m(\mathfrak{p}, w) = t - 2 \times \#\{(\beta_i; j) \in \mathfrak{p} \,:\, w\beta_i < 0\}. \tag{3.10}$$

With $\mathfrak{p}$ as in equation (3.9), we define

$$\phi_i(\mathfrak{p}) := \begin{cases} \{(s_i\beta_1; m_1), \ldots, (s_i\beta_t; m_t), (\alpha_i; 1)\}, & \text{if } \beta_j \neq \alpha_i \text{ for any } j, \\ \{(s_i\beta_1; m_1), \ldots, (s_i\beta_{j-1}; m_{j-1}), (s_i\beta_{j+1}; m_{j+1}), \ldots, (s_i\beta_t; m_t)\}, & \text{if } \beta_j = \alpha_i \text{ for some } j. \end{cases}$$

Since

$$-s_i \circ \lambda = -s_i(\lambda + \rho) + \rho = -s_i(\lambda) + \alpha_i,$$

we see that $\phi_i(\mathfrak{p}) \in \mathcal{P}(-s_i \circ \lambda)$. In other words, ϕ_i defines a map $\mathcal{P}(-\lambda) \to \mathcal{P}(-s_i \circ \lambda)$. Replacing λ with $s_i \circ \lambda$, we obtain similarly a map from $\mathcal{P}(-s_i \circ \lambda)$ to $\mathcal{P}(-\lambda)$. One can check that these maps are inverses to each other, and so the map ϕ_i is a bijection from $\mathcal{P}(-\lambda)$ to $\mathcal{P}(-s_i \circ \lambda)$.

Lemma 3.9. *If $\ell(ws_i) = \ell(w) + 1$, then*

$$m(\phi_i(\mathfrak{p}), w) = m(\mathfrak{p}, ws_i) + 1.$$

Proof. Consider $\phi_i(\mathfrak{p}) = \{(\beta_1; m_1), \dots, (\beta_t; m_t)\} \in \mathcal{P}(-\lambda)$. First assume that

$$\mathfrak{p} = \{(s_i\beta_1; m_1), \dots, (s_i\beta_t; m_t), (\alpha_i; 1)\}.$$

By applying ws_i to the first components, we get $w\beta_1, \dots, w\beta_t, -w\alpha_i$. Since $-w\alpha_i < 0$ from the condition $\ell(ws_i) = \ell(w) + 1$, we obtain $m(\mathfrak{p}, ws_i) = m(\phi_i(\mathfrak{p}), w) + 1 - 2 = m(\phi_i(\mathfrak{p}), w) - 1$. Next assume that

$$\mathfrak{p} = \{(s_i\beta_1; m_1), \dots, (s_i\beta_{j-1}; m_{j-1}), (s_i\beta_{j+1}; m_{j+1}), \dots, (s_i\beta_t; m_t)\}.$$

In this case, we have $\beta_j = \alpha_i$, and obtain $m(\mathfrak{p}, ws_i) = m(\phi_i(\mathfrak{p}), w) - 1$. □

Proposition 3.10. *If $\lambda \in Q^-$ and $w \in W$, then*

$$(-1)^{\ell(w)} H(-w \circ \lambda) = q^{\ell(w)} \sum_{\mathfrak{p} \in \mathcal{P}(-\lambda)} (-q)^{m(\mathfrak{p}, w)}. \tag{3.11}$$

Proof. Write $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ as a reduced expression. By Lemma 3.9, we have

$$\begin{aligned} H(-s_{i_1} s_{i_2} \cdots s_{i_\ell} \circ \lambda) &= \sum_{\mathfrak{p} \in \mathcal{P}(-s_{i_1} s_{i_2} \cdots s_{i_\ell} \circ \lambda)} (-q)^{m(\mathfrak{p}, \mathrm{id})} \\ &= \sum_{\mathfrak{p} \in \mathcal{P}(-s_{i_2} \cdots s_{i_\ell} \circ \lambda)} (-q)^{m(\phi_{s_{i_1}}(\mathfrak{p}), \mathrm{id})} = \sum_{\mathfrak{p} \in \mathcal{P}(-s_{i_2} \cdots s_{i_\ell} \circ \lambda)} (-q)^{m(\mathfrak{p}, s_{i_1}) + 1} \\ &= \sum_{\mathfrak{p} \in \mathcal{P}(-s_{i_3} \cdots s_{i_\ell} \circ \lambda)} (-q)^{m(\phi_{s_{i_2}}(\mathfrak{p}), s_{i_1})} = \sum_{\mathfrak{p} \in \mathcal{P}(-s_{i_3} \cdots s_{i_\ell} \circ \lambda)} (-q)^{m(\mathfrak{p}, s_{i_1} s_{i_2}) + 2} \\ &= \cdots = \sum_{\mathfrak{p} \in \mathcal{P}(-\lambda)} (-q)^{m(\mathfrak{p}, s_{i_1} \cdots s_{i_\ell}) + \ell}, \end{aligned}$$

which amounts to the identity (3.11). □

4. Polynomiality

In this section we prove Theorem 1.2. That is, we show that d_λ is a polynomial when the Weyl group W of $\mathfrak{g}$ is a universal Coxeter group.

Assume that W be a universal Coxeter group of rank $n \in \mathbb{Z}_{>0}$. By definition, the group W is isomorphic to the free product of n-copies of $\mathbb{Z}/2\mathbb{Z}$. Denote its generators by s_i, $i = 1, \dots, n$. The identity element is the only word of length 0, and for any $\ell \geq 1$ there are $n(n-1)^{\ell-1}$ words of length ℓ. We thus compute the Poincaré series of W to be:

$$\chi(q) = \sum_{w \in W} q^{\ell(w)} = 1 + nq \sum_{k=0}^{\infty} ((n-1)q)^k = 1 + \frac{nq}{1-(n-1)q} = \frac{1+q}{1-(n-1)q} \in \mathbb{Z}[[q]].$$

Given $\lambda \in P^+ \cap Q^-_{\mathrm{im}}$, our aim is to establish the polynomiality of

$$d_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} H(w \circ \lambda)}{\chi(q)} = \frac{(1-(n-1)q) \sum_{w \in W} (-1)^{\ell(w)} H(w \circ \lambda)}{1+q} \in \mathbb{Z}[[q]]. \tag{4.1}$$

Fix $\lambda \in P^+ \cap Q^-_{\mathrm{im}}$, and define

$$N = \max\{\ell(w) + 1 : (w\alpha_i; j) \text{ is a part of any } \mathfrak{p} \in \mathcal{P}(-\lambda) \text{ for } 1 \leq i \leq n,\ j \in \mathbb{Z} \text{ and } w \in W\}. \tag{4.2}$$

Since W is a universal Coxeter group, the number N is well-defined. For the time being, fix an arbitrary element $v \in W$ of length N. Let W_v be the set of elements in W whose reduced word has v as its rightmost factor. For $\mathfrak{p} \in \mathcal{P}(-\lambda)$, define

$$m(\mathfrak{p}, W_v) = |\mathfrak{p}| - 2 \times \#\left\{(\beta_i; j) \in \mathfrak{p} \,:\, w\beta_i < 0 \text{ for some } w \in W_v\right\}. \tag{4.3}$$

Write

$$\sum_{\mathfrak{p}\in\mathcal{P}(-\lambda)} (-q)^{m(\mathfrak{p},W_v)} = \sum_{k=0}^{r} a_k q^k \tag{4.4}$$

for some $a_k \in \mathbb{Z}$ and $r \geq 0$, and define

$$Q_v := \sum_{k=0}^{r-1} \Big((n-1)^k a_0 + (n-1)^{k-1} a_1 + (n-1)^{k-2} a_2 + \cdots + a_k\Big)\, q^k, \tag{4.5}$$

$$A_v := (n-1)^r a_0 + (n-1)^{r-1} a_1 + (n-1)^{r-2} a_2 + \cdots + a_r. \tag{4.6}$$

It follows from (3.11) that

$$\begin{aligned}\sum_{w\in W_v} (-1)^{\ell(w)} H(-w\circ\lambda) &= \sum_{w\in W_v} q^{\ell(w)} \sum_{\mathfrak{p}\in\mathcal{P}(-\lambda)} (-q)^{m(\mathfrak{p},w)} \\ &= q^N \left(Q_v + A_v\, q^r \left(1 + (n-1)q + (n-1)^2 q^2 + \cdots\right)\right) \\ &= q^N \left(Q_v + A_v \frac{q^r}{1-(n-1)q}\right),\end{aligned}$$

and so

$$\begin{aligned}&\sum_{w\in W} (-1)^{\ell(w)} H(-w\circ\lambda) \\ &\qquad = \sum_{w\in W, \ell(w)<N} q^{\ell(w)} \sum_{\mathfrak{p}\in\mathcal{P}(-\lambda)} (-q)^{m(\mathfrak{p},w)} + q^N \sum_{v\in W, \ell(v)=N} \left(Q_v + A_v \frac{q^r}{1-(n-1)q}\right).\end{aligned} \tag{4.7}$$

Proposition 4.1. *For any $w \in W$, the sum*

$$\sum_{w\in W} (-1)^{\ell(w)} H(-w\circ\lambda)$$

is divisible by $1+q$.

Proof. Let N be defined as in (4.2), and $v \in W$ be an arbitrary element of length N. For any $w \in W$, we have

$$\sum_{\mathfrak{p}\in\mathcal{P}(-\lambda)} 1^{m(\mathfrak{p},w)} = |\mathcal{P}(-\lambda)|.$$

Therefore

$$\sum_{\substack{w\in W\\ \ell(w)<N}} (-1)^{\ell(w)} \sum_{\mathfrak{p}\in\mathcal{P}(-\lambda)} 1^{m(\mathfrak{p},w)} = |\mathcal{P}(-\lambda)| \sum_{\substack{w\in W\\ \ell(w)<N}} (-1)^{\ell(w)}.$$

Since

$$\sum_{\substack{w\in W\\ \ell(w)<N}} (-1)^{\ell(w)} = \sum_{k=0}^{N-1} (-1)^k \#\{w \in W : \ell(w) = k\},$$

and

$$\#\{w \in W : \ell(w) = k\} = \begin{cases} 1, & k = 0 \\ n(n-1)^{k-1}, & k > 0, \end{cases}$$

we deduce

$$\begin{aligned}\sum_{\substack{w\in W\\ \ell(w)<N}} (-1)^{\ell(w)} &= 1 - n + n(n-1) - n(n-1)^2 + \cdots + (-1)^{N-1} n(n-1)^{N-2} \\ &= 1 - n\left[1 + (-1)(n-1) + (-1)^2(n-1)^2 + \cdots + (-1)^{N-2}(n-1)^{N-2}\right] \\ &= 1 - n\left[1 + (1-n) + (1-n)^2 + \cdots + (1-n)^{N-2}\right] \\ &= 1 - n\left(\frac{1-(1-n)^{N-1}}{1-(1-n)}\right) = (1-n)^{N-1}.\end{aligned}$$

Combining the above, we see that

$$\sum_{\substack{w\in W\\ \ell(w)<N}} (-1)^{\ell(w)} \sum_{\mathfrak{p}\in\mathcal{P}(-\lambda)} 1^{m(\mathfrak{p},w)} = (1-n)^{N-1}|\mathcal{P}(-\lambda)|.$$

Let A_v and Q_v be defined as in (4.6) and (4.5), respectively. Then we have

$$\frac{(-1)^r}{n} A_v = \frac{(-1)^r}{n}\left((n-1)^r a_0 + (n-1)^{r-1} a_1 + \cdots + a_r\right),$$

and

$$\begin{aligned}Q_v = a_0&\left(1 - (n-1) + (n-1)^2 + \cdots + (-1)^{r-1}(n-1)^{r-1}\right) \\ &- a_1\left(1 - (n-1) + (n-1)^2 + \cdots + (-1)^{r-2}(n-1)^{r-2}\right) + \cdots + (-1)^{r-1} a_{r-1} \\ &= \tfrac{1-(1-n)^r}{n} a_0 - \tfrac{1-(1-n)^{r-1}}{n} a_1 + \cdots + (-1)^{r-2}\tfrac{1-(1-n)^2}{n} a_{r-2} + (-1)^{r-1} a_{r-1},\end{aligned}$$

so that

$$Q_v + A_v\frac{(-1)^r}{n} = \frac{1}{n}\left(a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^r a_r\right) = \frac{1}{n}|\mathcal{P}(-\lambda)|.$$

Evaluating (4.7) at $q = -1$, we get

$$(1-n)^{N-1}|\mathcal{P}(-\lambda)| + (-1)^N n(n-1)^{N-1}\frac{1}{n}|\mathcal{P}(-\lambda)| = 0.$$

□

Proof of Theorem 1.2. It follows from (4.7) that

$$(1-(n-1)q)\sum_{w\in W}(-1)^{\ell(w)} H(-w\circ\lambda)$$

is a polynomial. By Proposition 4.1, the sum $\sum_{w\in W}(-1)^{\ell(w)} H(-w\circ\lambda)$ is divisible by $1+q$. Thus we see from (4.1) that d_λ is a polynomial. □

Remark 4.2. From [KL12, (3-21)], we know that

$$H(-w\circ\lambda;-1) = H(\rho - w(\lambda+\rho);-1) = \dim V(\rho)_{w(\lambda+\rho)} = \dim V(\rho)_{\lambda+\rho}. \tag{4.8}$$

Taking the alternating sum, we get

$$\sum_{w\in W}(-1)^{\ell(w)} H(-w\circ\lambda;-1) = \dim V(\rho)_{\lambda+\rho}\sum_{w\in W}(-1)^{\ell(w)},$$

which does not converge. In Proposition 4.1, the sum $\sum_{w\in W}(-1)^{\ell(w)}H(-w\circ\lambda;q)$ is to be interpreted via its analytic continuation given by the rational function in equation (4.7).

Appendix A.

In this appendix, we consider the explicit example of the Kac–Moody algebra $\mathfrak{g}=\mathcal{H}(3)$ associated to the generalized Cartan matrix

$$A=\begin{pmatrix}2 & -3\\ -3 & 2\end{pmatrix}. \tag{A.1}$$

The Weyl group W is the universal Coxeter group of rank 2, that is, W is isomorphic to the free product $(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/2\mathbb{Z})$. As there are two elements for a given length ≥ 1, the Poincaré series has the following closed form:

$$\chi(q)=1+2q\sum_{\ell=0}^{\infty}q^{\ell}=1+2q\left(\frac{1}{1-q}\right)=\frac{1+q}{1-q}\in\mathbb{Z}[[q]].$$

We denote the simple roots by α_1,α_2 and the simple reflections by s_1,s_2 as before. When $\mathfrak{p}=\{(\beta_1;m_1),(\beta_2;m_2),\dots,(\beta_t;m_t)\}$ is an admissible partition, we will sometimes write

$$\mathfrak{p}=(\beta_1;m_1)+(\beta_2;m_2)+\cdots+(\beta_t;m_t).$$

For the root multiplicities of $\mathcal{H}(3)$, we refer the reader to [KaMe].

Example A.1. Consider $\lambda=-2\alpha_1-2\alpha_2\in Q^-$. Then there are 4 admissible partitions of $-\lambda$:

(1) $(2\alpha_1+2\alpha_2;1)$,
(2) $(\alpha_1;1)+(\alpha_1+2\alpha_2;1)$,
(3) $(\alpha_2;1)+(2\alpha_1+\alpha_2;1)$,
(4) $(\alpha_1;1)+(\alpha_2;1)+(\alpha_1+\alpha_2;1)$.

All the roots appearing in the list above have multiplicity 1, and so

$$H(-\lambda)=H(2\alpha_1+2\alpha_2)=-q+2q^2-q^3=-q(q-1)^2.$$

We calculate

$$-s_1\circ\lambda=-s_1(\lambda+\rho)+\rho=5\alpha_1+2\alpha_2,$$

and see that $-s_1\circ\lambda$ has 4 admissible partitions:

(1) $(5\alpha_1+2\alpha_2;1)$,
(2) $(\alpha_1;1)+(4\alpha_1+2\alpha_2;1)$,
(3) $(2\alpha_1+\alpha_2;1)+(3\alpha_1+\alpha_2;1)$,
(4) $(\alpha_1;1)+(\alpha_1+\alpha_2;1)+(3\alpha_1+\alpha_2;1)$.

Again, all the roots appearing have multiplicity 1. We therefore deduce that

$$H(-s_1\circ\lambda)=-q(q-1)^2=H(-\lambda).$$

Similarly, we compute

$$H(-s_2\circ\lambda)=-q(q-1)^2=H(-\lambda).$$

The circle action of s_1s_2 on λ yields

$$-s_1s_2\circ\lambda=14\alpha_1+5\alpha_2,$$

which is not a root. Yet again we have 4 admissible partitions, but the lengths are different:

(1) $(\alpha_1;1)+(13\alpha_1+5\alpha_2;1)$,

(2) $(\alpha_1;1)+(3\alpha_1+\alpha_2;1)+(10\alpha_1+4\alpha_2;1)$,
(3) $(\alpha_1;1)+(5\alpha_1+2\alpha_2;1)+(8\alpha_1+3\alpha_2;1)$,
(4) $(\alpha_1;1)+(2\alpha_1+\alpha_2;1)+(3\alpha_1+\alpha_2;1)+(8\alpha_1+3\alpha_2;1)$,

in which all the roots still have multiplicity 1. It follows that

$$H(-s_1s_2\circ\lambda)=q^2(q-1)^2=-qH(-s_1\circ\lambda)=-qH(-\lambda).$$

One can see that this pattern continues as proved in the previous section to yield

$$H(-w\circ\lambda)=(-q)^{\ell(w)-1}H(-\lambda),\quad w\in W,\ w\neq \mathrm{id}. \tag{A.2}$$

It follows from equations (3.3) and (A.2) that

$$\begin{aligned}\sum_{w\in W}(-1)^{\ell(w)}H(-w\circ\lambda)&=H(-\lambda)+\sum_{w\in W,\ w\neq\mathrm{id}}(-1)^{\ell(w)}(-q)^{\ell(w)-1}H(-\lambda)\\&=(1+q^{-1})H(-\lambda)-q^{-1}\chi(q)H(-\lambda),\end{aligned}$$

and so

$$d_{-2\alpha_1-2\alpha_2}=\left[(1+q^{-1})\frac{1-q}{1+q}-q^{-1}\right]H(-\lambda)=-H(-\lambda)=q(q-1)^2.$$

Example A.2. Let $\lambda=-2\alpha_1-3\alpha_2\in Q^-$, which is a root with multiplicity 2. We have admissible partitions:

(1) $(2\alpha_1+3\alpha_2,n)$, $n\in\{1,2\}$,
(2) $(2\alpha_1+2\alpha_2,1)+(\alpha_2,1)$,
(3) $(\alpha_1+3\alpha_2,1)+(\alpha_1,1)$,
(4) $(\alpha_1+2\alpha_2,1)+(\alpha_1+\alpha_2,1)$,
(5) $(\alpha_1+2\alpha_2,1)+(\alpha_1,1)+(\alpha_2,1)$.

Therefore

$$H(-\lambda)=-2q+3q^2-q^3=-q(q-1)(q-2).$$

We have

$$-s_1\circ\lambda=8\alpha_1+3\alpha_2,$$

which is a root with multiplicity 1, and admissible partitions:

(1) $(8\alpha_1+3\alpha_2,1)$,
(2) $(7\alpha_1+3\alpha_2,n)+(\alpha_1,1)$, $n\in\{1,2\}$,
(3) $(5\alpha_1+2\alpha_2,1)+(3\alpha_1+\alpha_2,1)$,
(4) $(5\alpha_1+2\alpha_2,1)+(2\alpha_1+\alpha_2,1)+(\alpha_1,1)$,
(5) $(4\alpha_1+2\alpha_2,1)+(3\alpha_1+\alpha_2,1)+(\alpha_1,1)$.

Therefore

$$H(-s_1\circ\lambda)=-q+3q^2-2q^3=-q(q-1)(2q-1).$$

On the other hand

$$-s_2\circ\lambda=2\alpha_1+4\alpha_2,$$

which is a root of multiplicity 1, and admissible partitions:

(1) $(2\alpha_1+4\alpha_2,1)$,
(2) $(2\alpha_1+3\alpha_2,n)+(\alpha_2,1)$, $n\in\{1,2\}$,
(3) $(\alpha_1+3\alpha_2,1)+(\alpha_1+\alpha_2,1)$,
(4) $(\alpha_1+3\alpha_2,1)+(\alpha_1,1)+(\alpha_2,1)$,
(5) $(\alpha_1+2\alpha_2,1)+(\alpha_1+\alpha_2,1)+(\alpha_2,1)$.

Therefore:

$$H(-s_2 \circ \lambda) = -q + 3q^2 - 2q^3.$$

Now

$$-s_2 s_1 \circ \lambda = 8\alpha_1 + 22\alpha_2,$$

which is not a root. We make a list of all admissible partitions:

(1) $(8\alpha_1 + 21\alpha_2, 1) + (\alpha_2, 1)$,
(2) $(7\alpha_1 + 18\alpha_2, n_1) + (\alpha_1 + 3\alpha_2, 1) + (\alpha_2, 1)$, $n_1 \in \{1, 2\}$,
(3) $(5\alpha_1 + 13\alpha_2, 1) + (3\alpha_1 + 8\alpha_2, 1) + (\alpha_2, 1)$,
(4) $(5\alpha_1 + 13\alpha_2, 1) + (2\alpha_1 + 5\alpha_2, 1) + (\alpha_1 + 3\alpha_2, 1) + (\alpha_2, 1)$,
(5) $(4\alpha_1 + 10\alpha_2, 1) + (3\alpha_1 + 8\alpha_2, 1) + (\alpha_1 + 3\alpha_2, 1) + (\alpha_2, 1)$.

It follows that

$$H(-s_2 s_1 \circ \lambda) = q^2 - 3q^2 + 2q^4 = q^2(2q^2 - 3q + 2).$$

On the other hand,

$$-s_1 s_2 \circ \lambda = 11\alpha_1 + 4\alpha_2,$$

which is not a root, and its admissible partitions are:

(1) $(10\alpha_1 + 4\alpha_2, 1) + (\alpha_1, 1)$,
(2) $(8\alpha_1 + 3\alpha_2, 1) + (3\alpha_1 + \alpha_2, 1)$,
(3) $(8\alpha_1 + 3\alpha_2, 1) + (2\alpha_1 + \alpha_2, 1) + (\alpha_1, 1)$,
(4) $(7\alpha_1 + 3\alpha_2, n) + (3\alpha_1 + \alpha_2) + (\alpha_1, 1)$, $n \in \{1, 2\}$,
(5) $(5\alpha_1 + 2\alpha_2, 1) + (3\alpha_1 + 2\alpha_2, 1) + (2\alpha_1 + \alpha_2, 1) + (\alpha_1, 1)$.

Therefore:

$$H(-s_1 s_2 \circ \lambda) = 2q^2 - 3q^3 + q^4 = q^2(q - 1)(q - 2).$$

Next

$$-s_1 s_2 s_1 \circ \lambda = 59\alpha_1 + 22\alpha_2,$$

which is not a root. We have

$$H(-s_1 s_2 s_1 \circ \lambda) = -q^3 + 3q^4 - 2q^5.$$

Also

$$-s_2 s_1 s_2 \circ \lambda = 11\alpha_1 + 30\alpha_2,$$

and

$$H(-s_2 s_1 s_2 \circ \lambda) = -2q^3 + 3q^4 - q^5.$$

We arrange the information above into a table, in which the columns are indexed by $n \in \mathbb{N}$ and the rows are indexed by $w \in W$ (written as reduced words, ordered lexicographically). The entry corresponding to row w and column n is the coefficient of q^n in $H(-w \circ \lambda)$. An empty space indicates that the coefficient is zero. There is one additional column, which lists the image $w \circ \lambda$ of λ under the circle action by w, written in coordinates with respect to the basis $\{-\alpha_1, -\alpha_2\}$.

w	$w \circ \lambda$	1	2	3	4	5	$\cdots$
id	$(2,3)$	-2	3	-1			
s_1	$(8,3)$	-1	3	-2			
s_2	$(2,4)$	-1	3	-2			
s_2s_1	$(8,22)$		1	-3	2		
s_1s_2	$(11,4)$		2	-3	1		
$s_1s_2s_1$	$(59,22)$			-1	3	-2	
$s_2s_1s_2$	$(11,30)$			-2	3	-1	
$\vdots$	$\vdots$				$\vdots$	$\vdots$	$\ddots$

Observe that the strings $(-2,3,-1)$ and $(-1,3,-2)$ repeat with each iteration, shifting 1 space and switching signs as the word length increases. The coefficient of q^n in $\chi(q)\,d_\lambda$ can be calculated by taking the sum of the entries in a column multiplied by $(-1)^{\ell(w)}$. We see that

$$d_{-2\alpha_1-3\alpha_2} = 0.$$

Example A.3. Consider $\lambda = -3\alpha_1 - 3\alpha_2 \in Q^-$. There are 12 admissible partitions of $-\lambda$:

(1) $(3\alpha_1 + 3\alpha_2; 1)$,
(2) $(3\alpha_1 + 3\alpha_2; 2)$,
(3) $(3\alpha_1 + 3\alpha_2; 3)$,
(4) $(3\alpha_1 + 2\alpha_2; 1) + (\alpha_2; 1)$,
(5) $(3\alpha_1 + 2\alpha_2; 2) + (\alpha_2; 1)$,
(6) $(2\alpha_1 + 3\alpha_2; 1) + (\alpha_1; 1)$,
(7) $(2\alpha_1 + 3\alpha_2; 2) + (\alpha_1; 1)$,
(8) $(2\alpha_1 + 2\alpha_2; 1) + (\alpha_1 + \alpha_2; 1)$,
(9) $(2\alpha_1 + \alpha_2; 1) + (\alpha_1 + 2\alpha_2; 1)$,
(10) $(\alpha_1; 1) + (\alpha_1 + \alpha_2; 1) + (\alpha_1 + 2\alpha_2; 1)$,
(11) $(\alpha_2; 1) + (\alpha_1 + \alpha_2; 1) + (2\alpha_1 + \alpha_2; 1)$,
(12) $(\alpha_1; 1) + (\alpha_2; 1) + (2\alpha_1 + 2\alpha_2; 1)$.

Note that $m(3\alpha_1 + 3\alpha_2) = 3$, $m(2\alpha_1 + 3\alpha_2) = m(3\alpha_1 + 2\alpha_2) = 2$, and the other roots each have multiplicity 1. We conclude

$$H(-\lambda) = H(3\alpha_1 + 3\alpha_2) = -3q + 6q^2 - 3q^3 = -3q(q-1)^2.$$

We continue to obtain the following table.

w	$w \circ \lambda$	1	2	3	4	5	6	$\cdots$
id	(3,3)	-3	6	-3				
s_1	(7,3)	-2	5	-4	1			
s_2	(3,7)	-2	5	-4	1			
s_1s_2	(19,7)		2	-5	4	-1		
s_2s_1	(7,19)		2	-5	4	-1		
$s_1s_2s_1$	(19,51)			-2	5	-4	1	
$s_2s_1s_2$	(51,19)			-2	5	-4	1	
$\vdots$	$\vdots$				$\vdots$	$\vdots$	$\vdots$	$\ddots$

Observe that the string $(-2,5,-4,1)$ repeats with each iteration, shifting 1 space and switching signs as the word length increases. Since $-2+5-4+1=0$, the coefficient of q^n is 0 for $n \geq 4$. As it happens, the coefficient of q^2 is 0 too. We conclude

$$d_{-3\alpha_1-3\alpha_2} = \frac{-q^3+q}{\chi(q)} = q(q-1)^2.$$

Example A.4. Let $\lambda = -3\alpha_1 - 4\alpha_2 \in Q^-$. We produce a table similar to that in Example A.3

w	$w \circ \lambda$	1	2	3	4	5	6	7	$\cdots$
id	$(3,4)$	-4	8	-5	1				
s_1	$(10,4)$	-1	7	-8	2				
s_2	$(3,6)$	-3	8	-6	1				
s_2s_1	(10,27)		1	-7	8	-2			
s_1s_2	(16,6)		4	-9	5				
$s_1s_2s_1$	(72,27)			-1	7	-8	2		
$s_2s_1s_2$	(16,43)			-4	9	-5			
$s_1s_2s_1s_2$	(72,190)				1	-7	8	-2	
$s_2s_1s_2s_1$	(114,43)				4	-9	5		
$\vdots$	$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$

This time, the strings $(1,-7,8,-2)$ and $(4,-9,5)$ alternate. Note that both strings sum to zero. We see that for $n \geq 5$, the coefficient of q^n in d_λ is 0. The coefficients of q and q^4 are also 0. Altogether we obtain

$$d_{-3\alpha_1-4\alpha_2} = \frac{-2q^2(1+q)}{\chi(q)} = 2q^2(q-1).$$

Example A.5. We may compute other d_λ's in a similar way. In the following table, the entry in the space (m,n) is the polynomial d_λ for $\lambda = -m\alpha_1 - (m+n)\alpha_2 \in P^+$. From symmetry in $\mathcal{H}(3)$, we have $d_{-m\alpha_1-(m+n)\alpha_2} = d_{-(m+n)\alpha_1-m\alpha_2}$.

	0	1	2	3	4
0	1	$-q(q-1)$	$q(q-1)^2$	$q(q-1)^2$	$2q(q-1)^2$
1			0	$2q^2(q-1)$	$-q^2(q-1)(q-4)$
2					$-q(q-1)^2(q^2+q-1)$

We also have

$$d_{-5\alpha_1-5\alpha_2} = -q(q-1)(q^3+3q^2-7q+2).$$

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