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MULTIVARIATE APPROXIMATIONS TO PORTFOLIO RETURN DISTRIBUTION

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Abstract

This article proposes a three-step procedure to estimate portfolio return distributions under the multivariate Gram-Charlier (MGC) distribution. The method combines quasi maximum likelihood (QML) estimation for conditional means and variances and the method of moments (MM) estimation for the rest of the density parameters, including the correlation coefficients. The procedure involves consistent estimates even under density misspecification and solves the so-called 'curse of dimensionality' of multivariate modelling. Furthermore, the use of a MGC distribution represents a flexible and general approximation to the true distribution of portfolio returns and accounts for all its empirical regularities. An application of such procedure is performed for a portfolio composed of three European indices as an illustration. The MM estimation of the MGC **(**MGC-MM**)** is compared with the traditional maximum likelihood of both the MGC and multivariate Student's t (benchmark) densities. A simulation on Value-at-Risk (VaR) performance for an equally weighted portfolio at 1% and 5% confidence indicates that the MGC-MM method provides reasonable approximations to the true empirical VaR. Therefore, the procedure seems to be a useful tool for risk managers and practitioners.

Keywords: European stock indices; Gram-Charlier expansion; Method of moments; Portfolio returns.

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1. Introduction

During the last decades, the literature related to the search of statistical models to explain and forecast financial risk has undergone huge developments. The interest derives from the needs of risk managers of financial institutions who must decide on the most appropriate model for portfolio and risk management. For these purposes many perspectives have been proposed mainly concerning either the modelling of the conditional moment structure under normality or the underlying distribution of the asset returns.

Among the latter approach one of the most interesting and fruitful alternative has been the semi-nonparametric (SNP hereafter) methodology developed by authors such as, Sargan (1975), Jarrow and Rudd (1982), Gallant and Nychka (1987), Gallant and Tauchen (1989), Corrado and Su (1997), Mauleón and Perote (2000), Nishiyama and Robinson (2000), Jondeau and Rockinger (2001), Velasco and Robinson (2001), Jurczenko et al. (2002), Verhoeven and McAleer (2004), Tanaka et al. (2005), León et al. (2005), Bao et al. (2006), Rompolis and Tzavalis (2006), León et al. (2009), Polanski and Stoja (2010), Ñíguez and Perote (2012) and Ñíguez et al. (2012). All these articles proposed the use of polynomial expansions of the Gaussian distribution to define density functions capable of capturing the stylized features of financial asset returns, besides of providing applications to the resulting densities to different contexts, e.g. hypotheses testing, density forecasting, Value-at-Risk (VaR hereafter), asset pricing or option valuation. The higher goodness-of-fit of this family of densities and the more accurate risk measures obtained, as shown in these papers, is a consequence of its more general and flexible representation, which admits as many parameters as necessary to capture the sharply-peaked, thick-tailed or skewed shapes of the underlying asset returns density.

These empirical findings emerge from the validity of the Gram-Charlier (GC hereafter) and Edgeworth series as asymptotic approximations – Charlier (1905) and Edgeworth (1907). As a matter of fact, under regular conditions, any frequency function can be expressed in terms of an infinite weighted sum of the derivatives of the standard Gaussian distribution or their corresponding Hermite polynomials (HP hereafter). The main shortcoming of these expansions is the fact that the positivity of the finite (truncated) expansions does not hold in the entire domain of the parameter set – Barton and Dennis (1952). This problem, has been tackled in the literature by means of parametric restrictions (Jondeau and Rockinger, 2001), or through density reformulations based on the methodology of Gallant and Nychka (1987). These solutions are not always the best option since imposing positivity constraints may lead to sub-optimization and positivity regions are not easy to be defined beyond the simpler cases (i.e. for expansions defined in terms of a couple of moments, usually skewness and kurtosis). Furthermore, positive transformations induce non-linearities among the distribution moments and the density parameters, and in some cases lead to symmetric distributions. The former problem affects the straightforward interpretation of the parameters of the raw GC density and thus seriously restricts the implementation of the method of moments (MM hereafter). Alternatively, Maximum Likelihood (ML hereafter) techniques are usually employed despite the fact that optimization algorithms usually fail to converge or do it to local optima. In addition, ML estimation only provides consistent estimates either under the normal or under the true density**.**

The extensions of GC densities to other continuous and differentiable non-normal densities have also been investigated. Particularly, the Poisson, Gamma, and Beta have been proposed as basis: GC Type B (Aroian, 1937), Laguerre (Muckenhoupt, 1969) and Jacobi (Szegö, 1975) expansions, respectively. Nevertheless, the validity of these series as asymptotic expansions and their empirical applicability are still to be proved (see Wallace, 1958, for a discussion on the validity of asymptotic expansions using non-normal densities as generating distributions). Generalizations of GC densities to the multivariate framework have also been proposed as alternatives to copula methods. In particular, Perote (2004) introduced a first definition and Del Brio et al. (2009; 2011) proposed more general formulations accounting for both the positivity and the 'curse of the dimensionality' problems, in the same spirit as the DCC model of Engle (2002). Recently, Ñíguez and Perote (2016) have applied these types of distributions to the DECO model by Engle and Kelly (2012).

This article revises the aforementioned multivariate models focusing on the implementation of a straightforward MM estimation as an alternative to the traditionally used ML or Quasi ML (QML hereafter) techniques. This proposal enhances the estimation algorithms of GC densities since the even (odd) parameters are just linear combinations of the even (odd) density moments and the moment of order n depends only on the first n density parameters. Even more, the MM estimation involves consistent estimates, which is only guaranteed for ML under the true density and for QML under density misspecification and provided that first and second moments are correctly specified (Bollerslev and Wooldridge, 1992). However, the current paper shows that all these techniques produce similar results and that a three-step estimation method can be straightforwardly implemented. The three-step method proceeds as follows: First, conditional variances under the normal distribution for every variable are estimated by QML; second, the rest of the GC density parameters for every variable are estimated by MM; third, the correlation among the portfolio variables is estimated by MM.

The remainder of the paper is structured as follows: Section 2 reviews the multivariate GC expansions and explains the MM estimation procedures. Section 3 provides an application of the MM technique for the estimation and VaR computation of the multivariate GC density of a portfolio composed of three European stock indices. The last section (4) summarizes the main conclusions of the paper.

2. Approximations to portfolio return distribution

This section revises the main results supporting GC series as a general approximation to a frequency function and some of the main statistical properties of the GC expansion that are the basis of the outstanding empirical performance of this approximation to account for portfolio return distributions. Section 2.1 presents the basic univariate (single asset) case and Section 2.2 generalises the results to a multivariate (portfolio) framework. In both cases the GC density is formulated with matrix notation to facilitate the comparison between both approaches.

2.1. The univariate case

Let $\phi(x_i) = e^{-x^2/2} \big/ \sqrt{2\pi}$ be the normal probability density function (pdf hereafter) and $H_s(x_i)$ the Hermite polynomial based on its s-th order derivative, which can be defined as in equation (1).

$$H_s(x_i) = (-1)^s \phi(x_i)^{-1} \frac{d^s \phi(x_i)}{dx_i^{\,s}} = s! \sum_{k=0}^{[s/2]} \frac{(-1)^k}{k!(s-2k)!2^k} x_i^{\,s-2k} \,. \tag{1}$$

These Hermite polynomials (HP hereafter) form an orthonormal basis. Thus, if $\mathrm{H}(x_i)' = \begin{bmatrix} H_1(x_i) & H_2(x_i) & \cdots & H_q(x_i) \end{bmatrix} \in \Re^q$ is the vector containing the first q HP then

$$\int \mathrm{H}(x_i)\mathrm{H}(x_i)'\phi(x_i)dx_i = \mathrm{S} = diag\{1!, 2!, \ldots, q!\}. \tag{2}$$

Furthermore, the vector of HP can be written as

$$\mathrm{H}(x_i) = \mathrm{B}\mathrm{Z}_i + \Gamma\mu, \tag{3}$$

where $\mathrm{Z}_i' = \begin{bmatrix} x_i & x_i^{\,2} & \cdots & x_i^{\,q} \end{bmatrix} \in \Re^q$, $\mu' = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_q \end{bmatrix} \in \Re^q$ is the vector containing the first q central moments of the normal distribution (i.e. $\mu_s = \dfrac{s!}{2(s-2)!}$ for s even, and zero otherwise), Γ is a $q \times q$ diagonal matrix capturing the sing of the intercepts for each HP, and B is the $q \times q$ matrix containing the coefficients of the variables in Z_i corresponding to every HP (without loss of generality Γ and B are defined considering q even).

$$\Gamma = diag\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots & (-1)^{q/2} \end{pmatrix}. \tag{4}$$

$$\mathrm{B}=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -3 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -6 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{q!}{2^{\frac{q-2}{2}}\frac{q-2}{2}2!} & \frac{q!}{2^{\frac{q-4}{2}}\frac{q-4}{2}4!} & \frac{q!}{2^{\frac{q-6}{2}}\frac{q-6}{2}6!} & \frac{q!}{2^{\frac{q-8}{2}}\frac{q-8}{2}8!} & \cdots & \frac{q!}{2(q-2)!} & 1 \end{bmatrix}. \tag{5}$$

One of the main advantages of this sequence of HP is the fact that under certain regular conditions (Cramér, 1925) a frequency function, $f(x_i)$, can be expanded formally in terms of GC Type A series, i.e.,

$$f(x_i)=\sum_{s=0}^{\infty}\delta_s\frac{d^s\phi(x_i)}{dx_i^s}=\sum_{s=0}^{\infty}(-1)^s\delta_s H_s(x_i)\phi(x_i), \tag{6}$$

where the δ_s coefficients,

$$\delta_s=\frac{(-1)^s}{s!}\int_{-\infty}^{\infty}f(x_i)H_s(x_i)dx_i, \tag{7}$$

measure the deviations of $f(x_i)$ from $\phi(x_i)$ and can also be expressed in terms of the (non-central) moments of the random variable x_i with pdf $f(x_i)$.

Nevertheless, for empirical purposes the asymptotic expansion needs to be truncated at a degree q and then the univariate GC density is defined as follows,

$$f_q(x_i,\mathrm{d}_i)=\left[1+\mathrm{H}(x_i)'\mathrm{d}_i\right]\phi(x_i), \tag{8}$$

where $\mathrm{d}_i'=\begin{bmatrix} d_{i1} & d_{i2} & \cdots & d_{iq}\end{bmatrix}\in\Re^q$ is a vector of parameters and, by convention, we consider $H_0(x_i)=1$ and $d_0=1$. This distribution in equation (8) satisfies interesting properties (see e.g. Mauleón and Perote, 2000). Among them, Proposition 1 enunciates the property in which the MM estimation method proposed in the present article is based.

Proposition 1: The first q moments of the GC distribution in equation (8) can be expressed as a linear function of the vector $\mathrm{d}_i\in\Re^q$,

$$E\left[Z_i\right]=\mathrm{B}^{-1}(\mathrm{S}\mathrm{d}_i-\Gamma\mu), \tag{9}$$

where $Z_i' = \begin{bmatrix} x_i & x_i^2 & \cdots & x_i^q \end{bmatrix}$, and S, Γ and B are the matrices described in equations (2), (4) and (5), respectively, and $\mu \in \Re^q$ is the vector containing the first q central moments of the normal distribution.

Proof:

$$E[Z_i] = \int Z[1 + H(x_i)'d_i]\phi(x_i)dx_i = \int B^{-1}(H(x_i) - \Gamma\mu)[1 + H(x_i)'d_i]\phi(x_i)dx_i$$

$$= B^{-1}\int H(x_i)\phi(x_i)dx_i + B^{-1}\int H(x_i)H(x_i)'d_i\phi(x_i)dx_i - B^{-1}\Gamma\mu\int \phi(x_i)dx_i - B^{-1}\Gamma\mu\int H(x_i)'d_i\phi(x_i)dx_i$$

$$= 0 + B^{-1}Sd_i - B^{-1}\Gamma\mu + 0 = B^{-1}(Sd_i - \Gamma\mu).$$

These relations among the density moments and parameters establish a straightforward way of estimating the density by the MM, given by

$$\hat{d}_i = S^{-1}(B\hat{E}[Z_i] + \Gamma\mu), \tag{10}$$

where $\hat{E}[Z_i]$ is the vector containing the first q sample moments of variable x_i with pdf $f_q(x_i, d_i)$.

The truncated function in (8), however, does not guarantee positivity for all values of d_i and thus a positive (squared) transformation of the Gallant and Nychka's (1987) type is usually implemented. Next we explain the family of multivariate GC densities including these positive transformations.

2.2. The multivariate case

A random vector $X' = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \in \Re^n$ belongs to the multivariate GC (MGC hereafter) family of distributions if it is distributed according to the following pdf,

$$F(X) = \frac{1}{n+1}\left[G(X) + \left\{\sum_{i=1}^{n}\frac{1}{c_i}h(x_i)'A_i h(x_i)\right\}\left\{\prod_{i=1}^{n}\phi(x_i)\right\}\right], \tag{11}$$

where

$$G(X) = (2\pi)^{-n/2}\left|\Sigma^{-1/2}\right|\exp\left\{-\frac{1}{2}X'\Sigma^{-1}X\right\} \tag{12}$$

is the multivariate normal pdf – with univarite marginals $\phi(x_i)$, A_i is a positive definite matrix of order (q+1), $\mathrm{h}(x_i)=\left[1 \quad \mathrm{H}(x_i)\right]\in\Re^{q+1}$ and $c_i=\int \mathrm{h}(x_i)'\mathrm{A}_i\mathrm{h}(x_i)\phi(x_i)dx_i$ ($\forall i$=1,2,…, n) are the constants that make the density integrating up to one, see Del Brio et al. (2009). Without loss of generality, the same truncation order q for every dimension i is assumed.

This MGC family encompasses many different distributions, such as the multivariate extensions of the GC density in León et al. (2009) or the Positive Edgeworth-Sargan in Ñíguez and Perote (2012). These two types of distributions are obtained by considering $A_i=D_iD_i'$ and $A_i=diag\left\{1,d_{i1}^2,...,d_{iq}^2\right\}$, respectively, where $\mathrm{D}_i=\left[1 \quad \mathrm{d}_i\right]\in\Re^{q+1}$ (note that in both cases $c_i=1+\sum_{s=1}^{q}d_{is}^2 s!$). However, in this paper we implement a related family of densities proposed in Perote (2004) which does not formally impose positive definiteness but presents other interesting advantages from an empirical perspective. Hereafter, we will refer to the pdf defined in equation (12) below as the MGC density.

$$F(\mathrm{X})=G(\mathrm{X})+\left\{\sum_{i=1}^{n}\mathrm{H}(x_i)'\mathrm{d}_i\right\}\left\{\prod_{i=1}^{n}\phi(x_i)\right\}. \tag{13}$$

It is clear that, for the MGC, the marginal density of x_i is that of equation (8) and thus the MM estimation can be trivially implemented through the relation in equation (10). Even more, Del Brio et al. (2011) proved that an equivalent MGC density can be estimated by ML in two steps: In the first step, the conditional mean and variance of every variable are estimated by QML independently, and in the second step, the rest of the density parameters are jointly estimated in the standardised distribution. This paper proposes a similar three-step procedure based on the MM: First, QML estimates for conditional mean and variance of every variable are obtained independently by assuming a normal distribution. Second, the parameters for the univariate GC density of every standardised variable are estimated independently by the MM. Third, correlation parameters are approximated by the sample correlations.

3. Empirical application

We illustrate the estimation procedure of the portfolio return distribution described in the previous section for a portfolio composed of three European stock indices: EUROSTOXX50, Ibex35 and Dax30. The sample comprises almost 10 years of daily data (T=2,861 observations) spanning from September 30th, 2002, to November 19th, 2013. We model continuously compounded returns, defined as

r_{it}=100log(P_{it}/$P_{i,t-1}$). Table 1 displays descriptive statistics for the series. These data feature the main empirical regularities of high-frequency financial returns: a small predictable component in the conditional mean, volatility clustering, skewness, leptokurtosis and, likely, multimodality (jumps) in the tails.

Table 1. Descriptive statistics for EUROSTOXX50, Ibex35 and Dax30.

	EUROSTOXX50	Ibex35	Dax30
Mean	0. 01134	0.02025	0.04228
Variance	2.21881	2.26692	2.26598
Minimum	-8.20788	-9.58586	-7.43346
Maximum	10.43765	13.48364	10.79747
Skewness	0.07902	0.13892	0.08702
Kurtosis	5.45580	6.52309	5.38219

We specify a multivariate AR(1)-GARCH(1,1) structure for modelling conditional first and second moments and the MGC density in equation (13) for capturing the rest of the salient empirical regularities of the data. Thus, the multivariate model for the portfolio returns $\mathrm{r_t}' = [r_{1t} \quad r_{2t} \quad r_{3t}] \in \Re^3$ is:

$$\mathrm{r_t} = \Phi_0 + \Phi_1' \mathrm{r_{t-1}} + \mathrm{u}_t, \tag{14}$$

$$\mathrm{u_t} | \Omega_{t-1} \approx MGC(0, \Sigma_t(\alpha, \rho)), \tag{15}$$

$$\Sigma_t(\alpha,\rho) = \mathrm{D}_t(\alpha)\mathrm{R}_t(\rho)\mathrm{D}_t(\alpha), \tag{16}$$

$$\mathrm{D}_t(\alpha)^2 = diag\{\alpha_{i0}\} + diag\{\alpha_{i1}\} \circ \mathrm{u}_{t-1}\mathrm{u}'_{t-1} + diag\{\alpha_{i2}\} \circ \mathrm{D}_{t-1}(\alpha)^2, \tag{17}$$

where $\Phi_0' = [\phi_{10} \quad \phi_{20} \quad \phi_{30}] \in \Re^3$ and $\Phi_1' = [\phi_{11} \quad \phi_{21} \quad \phi_{31}] \in \Re^3$ are the vectors containing the AR(1) models parameters; $diag\{\alpha_{i0}\} = diag(\alpha_{10} \quad \alpha_{20} \quad \alpha_{30})$, $diag\{\alpha_{i1}\} = diag(\alpha_{11} \quad \alpha_{21} \quad \alpha_{31})$ and $diag\{\alpha_{i2}\} = diag(\alpha_{12} \quad \alpha_{22} \quad \alpha_{32})$ are 3×3 diagonal matrices containing the parameters of the GARCH(1,1) processes (hereafter we refer to these parameters as Φ and α, respectively) and $\mathrm{u_t}' = [u_{1t} \quad u_{2t} \quad u_{3t}] \in \Re^3$ is a vector of model disturbances, which is assumed to be distributed according to the MGC pdf in equation (12) . Therefore, the variance and covariance matrix is decomposed in the diagonal matrix of conditional deviations, $\mathrm{D}_t(\alpha)$, and the symmetric correlation matrix, $\mathrm{R}_t(\rho)$, with general element $\{\rho_{ij}\}$ (hereafter we refer to the parameters in $\mathrm{R}_t(\rho)$ as ρ). Finally, $\circ$ is the Hadamard product of two identical sized matrices (computed by element-by-element multiplication).

The estimation of the model in equations (14)-(17) through our proposed three-step MM is carried out in the following three stages:

Stage 1: Φ and α are estimated by QML as the values that maximise the log-likelihood of every variable under the Gaussian distribution, i.e.,

$$\left\{\hat{\phi}_{i0}, \hat{\phi}_{i1}, \hat{\alpha}_{i0}, \hat{\alpha}_{i1}, \hat{\alpha}_{i2}\right\} = \arg\max\left\{-\frac{1}{2}\sum_{t=1}^{T}\left[\ln(\sigma_{it}^2) + \frac{(r_{it} - \phi_{i0} - \phi_{i1} r_{it-1})^2}{\sigma_{it}^2}\right]\right\}$$

s.t.

$$\sigma_{it}^2 = \alpha_{i0} + \alpha_{i1}(r_{it} - \phi_{i0} - \phi_{i1} r_{it-1})^2 + \alpha_{i2}\sigma_{it-1}^2;\ \forall i = 1,2,3. \quad (18)$$

Stage 2*:* The parameters (d_i) of the GC expansions are estimated independently for every dimension *i* by using the following correspondences,

$$\hat{d}_{i1} = \hat{m}_{i1}, \quad (19)$$

$$\hat{d}_{i2} = (\hat{m}_{i2} - 1)/2, \quad (20)$$

$$\hat{d}_{i3} = (\hat{m}_{i3} - 3\hat{m}_{i1})/6, \quad (21)$$

$$\hat{d}_{i4} = (\hat{m}_{i4} - 6\hat{m}_{i2} + 3)/24, \quad (22)$$

$$\hat{d}_{i5} = (\hat{m}_{i5} - 10\hat{m}_{i3} + 15\hat{m}_{i1})/120, \quad (23)$$

$$\hat{d}_{i6} = (\hat{m}_{i6} - 15\hat{m}_{i4} + 45\hat{m}_{i2} - 15)/720, \quad (24)$$

$$\hat{d}_{i7} = (\hat{m}_{i7} - 21\hat{m}_{i5} + 105\hat{m}_{i3} - 150\hat{m}_{i1})/5040, \quad (25)$$

$$\hat{d}_{i8} = (\hat{m}_{i8} - 28\hat{m}_{i6} + 210\hat{m}_{i4} - 420\hat{m}_{i2} + 150)/40320, \quad (26)$$

$\forall i = 1,2,3,$ where $\hat{m}_{li} = \frac{1}{T}\sum_{t=1}^{T}\frac{(r_{it} - \bar{r}_i)^l}{S_i^l}, \forall l = 1,2,...,8,$ is the *l*-th order sample moment of the standardised series ($\bar{r}_i$ and S_i denoting the average and standard deviation of r_{it}, respectively), which is a consistent estimate of the *l*-th order moment of the true distribution.

Stage 3: The Correlation matrix, R(ρ), is estimated by computing the sample correlations among the portfolio variables:

$$\hat{\rho}_{ij} = \frac{1}{TS_i S_j}\sum_{t=1}^{T}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j),\ \forall i, j = 1,2,3\ \ (i \neq j). \quad (27)$$

This method features the following advantages with respect to the ML: (i) It provides consistent estimates, i.e., the first step gives consistent (QML) estimates for conditional mean and variance parameters and the second step is also consistent since both log-likelihood function is separable (see Del Brio et al., 2011) and the MM always yields consistent estimates. (ii) It is much simpler than the ML method with regards to convergence problems that may arise in optimization. (iii) It solves the "curse of dimensionality" of multivariate modelling, since it is not affected by the number of assets considered in the portfolio. (iv) Parameter estimates are the same regardless the expansion length and, as it is shown empirically, the procedure leads to very similar outcomes for the estimated density than those obtained by ML.

Table 2 provides two-step ML estimates (t-ratios are displayed in parentheses) for the parameters of the GC density of a portfolio composed of EUROSTOXX50, Ibex35 and Dax30 indices. We consider expansions up to the eighth term but d_{i1} and d_{i2} are constrained to zero since conditional means and variances are captured by the AR(1) and the GARCH(1,1) models, respectively. The AR(1)-GARCH(1,1) parameters are estimated in the first step by QML. These QML estimates confirm the presence of a small predictable component in the conditional mean, and persistence and clustering in volatility (α_{i1}+α_{i2} is estimated close to one). In the second step, the rest of the parameters of the density are estimated by either MM or ML applied to the series standardised by the estimated mean and variance of the previous step. The estimates of the GC densities exhibit the traditional behaviour of stock returns: (i) (negative) skewness is captured by parameter d_{i3} and the rest of the odd parameters are not significant; (ii) leptokurtosis is patent since d_{i4} is positive and significant; and (iii) presence of extreme values as high order moments (parameters d_{i6} and d_{i8}) are also significant. Note that the truncation order is chosen according to accuracy criteria (see the Akaike information criteria, *AIC*, for the MGC model with 2 and 6 parameters), although the best model should eliminate the insignificant parameters that are still displayed in Table 2. Finally, the third stage presents the estimate for the correlation matrix, which exhibits a positive correlation between EUROSTOXX50 and both Ibex35 and Dax30, but absence of correlation between the latter two indices.

Table 2. MGC density of stock indices: EUROSTOXX50, Ibex35 and Dax30.

	EUROSTOXX50	Ibex35	Dax30
		***Stage* 1 (QML)**	
ϕ_{i1}	0.06776	0.08020	0.09300
	(3.290)	(3.865)	(4.409)
ϕ_{i2}	-0.05768	-0.00669	-0.03015
	(-2.938)	(-0.323)	(-1.569)
α_{i0}	0.02389	0.02018	0.02276
	(3.060)	(2.825)	(3.352)
α_{i1}	0.09721	0.09982	0.08949
	(5.265)	(5.078)	(6.452)
α_{i2}	0.89217	0.89359	0.89925
	(48.341)	(48.676)	(65.130)
		***Stage* 2 (MM)**	
d_{i3}	-0.04495	-0.04790	-0.05916
d_{i4}	0.05733	0.06572	0.04999
d_{i5}	-0.02341	-0.02207	-0.02759
d_{i6}	0.02333	0.02506	0.02251

d_{i7}	-0.01651	-0.01170	-0.01848
d_{i8}	0.01181	0.00959	0.01316
	***Stage* 2 (ML)**		
d_{i3}	-0.02260	-0.03181	-0.04127
	(-1.89038)	(-2.54818)	(-4.05200)
d_{i4}	0.04212	0.04937	0.02997
	(5.27076)	(5.66917)	(3.76514)
d_{i5}	0.00401	-0.00176	-0.00188
	(0.89059)	(-0.36192)	(-0.46320)
d_{i6}	0.00531	0.00808	0.00065
	(2.39301)	(3.25338)	(0.24688)
d_{i7}	0.00000	-0.00011	-0.00080
	(0.14397)	(-0.15316)	(-1.19259)
d_{i8}	0.00101	0.00102	0.00068
	(4.02314)	(3.79608)	(2.35141)
Loglikelihood	-1676.898	-1677.097	-1667.209
AIC (6 parameters)	3365.796	3366.194	3346.418
AIC (2 parameters)	3380.726	3380.912	3363.116
	***Stage* 3 (MM)**		
EUROSTOXX50	1	0.04727	0.03616
Ibex35	0.04727	1	0.00000
Dax30	0.03616	0.00000	1

t-statistics in parentheses

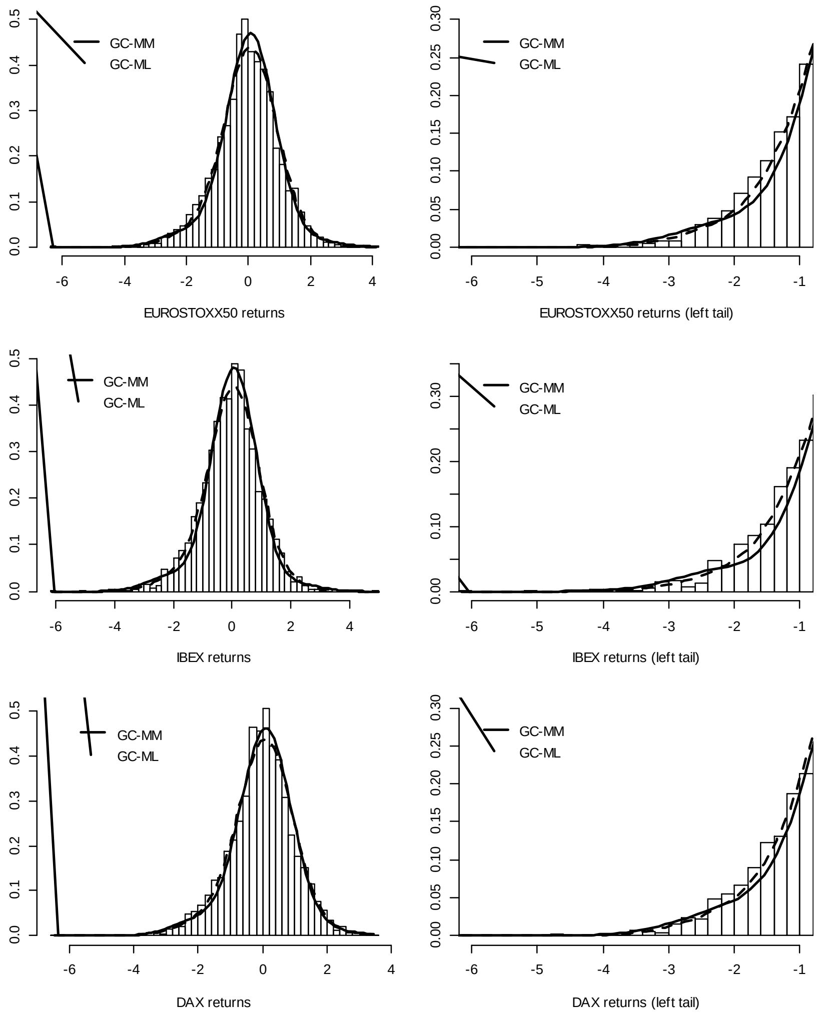
Table 3 shows the joint estimation for the multivariate Student's t distribution, the most common parametric alternative to account for non-normal portfolio returns. For this distribution, departures from normality are only captured by the degrees of freedom parameter (v) and thus it is a much restrictive estimation method.

Table 3. Multivariate t density for EUROSTOXX50, Ibex35 and Dax30.

	EUROSTOXX50	**Ibex35**	**Dax30**
ϕ_{i1}	0.06776	0.08020	0.09300
ϕ_{i2}	-0.05768	-0.00669	-0.03015
α_{i0}	0.02389	0.02018	0.02276
α_{i1}	0.09721	0.09982	0.08949
α_{i2}	0.89217	0.89359	0.89925
v		10.48116	
Loglikelihood		-11941.49	
		Correlation Matrix	
EUROSTOXX50	1	0.04284	0.04419
Ibex35	0.04284	1	-0.00065
Dax30	0.04419	-0.00065	1

Figure 1 depicts the fitted GC marginal distributions of the returns of the EUROSTOXX50, Ibex35 and Dax30 indices compared to the histogram of the data (non-parametric estimation). Figures on the left column represent the distributions for the whole range and figures on the right column the left tails (extreme values) of the corresponding distribution. The plots illustrate that both MM and ML methods (GC-MM and GC-ML, respectively) lead to very similar outcomes and that they approximate very accurately the empirical distribution of the portfolio. This evidence is even clearer in the tails of the distribution, which is the main focus of risk management.

Figure 1. Fitted GC distributions compared to the data histogram



Finally, we calculate VaR at 1% and 5% for an equally weighted portfolio formed with the three indices. For this purpose, 1,000 datasets of length 2,861 are simulated. Table 4 shows the average VaR and its standard error for the multivariate t, MGC estimated by MM and ML and the corresponding empirical VaR. The results illustrate how the MGC-MM model adequately captures portfolio VaR and thus **it** represents a very straightforward and useful method for risk management.

Table 4. Estimated VaR for an equally weighted portfolio of EUROSTOXX50, Ibex35 and Dax30.

	Multivariate t	**MGC-MM**	**MGC-ML**
Empirical VaR – 1%		-1.54395	
Mean VaR – 1%	-1.62432	-1.50203	-1.45894
Standard error	0.06022	0.05312	0.04867
Empirical VaR – 5%		-1.051891	
Mean VaR – 5%	-1.06971	-1.00847	-0.99632
Standard error	0.02996	0.02827	0.02711

The analyses performed in this section show not only that the GC densities result to be accurate approximations to portfolio distribution but also that the three-step MM procedure greatly simplifies their practical implementation. The former result was known from previous studies but the latter is a specific contribution of this paper. The main advantage of using the GC expansion to approximate the portfolio return distributions lies in its flexibility to improve data fits by adding more parameters. Nevertheless, it is noteworthy that the MM estimation is noisier the higher the order of the moments employed, what yields to efficiency losses compared to the traditional ML estimation. Notwithstanding, the three-step MM guarantees consistency even under misspecification and overcomes the curse of dimensionality, thus being specially recommended for large portfolios.

4. Conclusions

The GC density has emerged as a powerful tool to account for asset return distributions because it asymptotically captures the true distribution and thus represents a general and flexible approximation. Nevertheless, this distribution has scarcely been used for capturing the multivariate behaviour of portfolio distributions due to the so-called ‘curse of dimensionality’ that particularly affects this type of distributions that depend on a large number of parameters. Furthermore, the traditional ML estimation techniques usually fail to converge and, more importantly, do not guarantee consistency under possibly density misspecification. In order to solve these problems, this article proposes a three-step estimation method that combines QML estimation for conditional means and variances (Stage 1), MM estimation of the rest of the density parameters considering the univariate standardised marginal GC distributions and, finally, MM estimation of correlation coefficients.

The validity of this proposal is based on three main properties of the MGC distribution: (i) Its marginals behave as univariate GC distributions; (ii) it admits an independent estimation of the first and second moments under the Gaussian hypothesis (QML); and (iii) a direct linear relation exists among density moments and parameters, which simplifies the implementation of the MM techniques. Furthermore, this method is always consistent and may be straightforwardly implemented even for large portfolios. The main limitation of the MM technique is that the inclusion of high-order moments in large expansions might increase the estimation noise and reduce the efficiency of the estimates compared to ML.

An application of such procedure is performed for a portfolio composed of three European stock indices as an illustration of the method. The results are not very different from those obtained using QML estimation and thus it provides a straightforward method for estimating portfolio return distributions. The simplicity of the method as well as the asymptotic properties of the GC expansion makes this approach a very good approximation to portfolio distributions and thus an appealing methodology for risk managers. The application of this method to large portfolios and its comparison to recent techniques to estimate GC distributions in this framework (e.g. Ñíguez and Perote, 2016) might be an interesting avenue for future research.

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