Finance Research Letters 75 (2025) 106888

Contents lists available at ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

New bounds for tail risk measures

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ARTICLE INFO

Keywords:
Expected-shortfall
Kurtosis
Probability bound
Skewness
Value-at-risk

ABSTRACT

This paper introduces new upper bounds for tail risk measures, such as value-at-risk and expected shortfall, based on Bhattacharyya (1987) inequality. These enhanced bounds for losses consider higher-order moments like skewness and kurtosis, which sets them apart from the conventional one-sided Vysochanskii and Petunin (1980) and Cantelli (1928) inequalities. While the simplicity and reliance on estimating only the first two moments can make the latter bounds attractive, the practicality and effectiveness of the new bounds position them as a compelling alternative for risk measurement. We empirically analyze S&P 100 index stocks to illustrate our findings. Our results suggest tighter Basel multipliers and reduced minimum capital requirements.

1. Introduction

Probability bounds provide a conservative approach to estimating tail risk in financial returns, particularly when there is limited information on the first moments and no precise knowledge of probability distributions. Research, such as that by Barrieu and Scandolo (2015), has utilized these bounds to establish upper limits for tail risk measures (TRM) like value-at-risk (VaR) and expected shortfall (ES).

Some studies have also linked regulatory multipliers from Basel II and III to probability bounds for worst-case VaR scenarios. Barrieu and Scandolo (2015) and Kinateder (2016) noted that upper boundaries for VaR and ES can be derived using the one-sided Chebyshev (1867) or Cantelli (1928) inequality, with Basel multipliers falling in the [3,4] range, which has faced criticism for being too stringent during stable market periods; see Kinateder (2016).

Mercadier and Strobel (2021) introduced the one-sided Vysochanskii–Petunin (OSVP) inequality for unimodal return distributions, yielding tighter multiplier bounds for VaR in the [2,3] range. We apply the one-sided Bhattacharyya (Bhat) inequality to derive TRM upper bounds, incorporating higher-order moments like skewness and kurtosis. Our findings indicate that Bhat inequality results in more realistic TRM upper bounds and a tighter Basel regulatory multiplier than OSVP. Empirical results from modeling S&P 100 constituent losses support this. Moreover, Proposition 1 confirms the existence of VaR derived from Bhat inequality (Bhat-VaR), which can be calculated by solving a quartic equation, with one root relevant for identifying VaR for losses. Overall, while Cantelli (Cant) and OSVP bounds are simpler and rely only on the first two moments, Bhat bounds provide a more relevant approach for practitioners.

2. TRM bounds for losses

An implicit definition of VaR for the random variable (r.v.) of losses X at confidence level α is given by $\mathbb{P}(X \geq VaR_X(\alpha)) = \alpha$ where

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https://doi.org/10.1016/j.frl.2025.106888
Received 20 August 2024; Received in revised form 2 February 2025; Accepted 2 February 2025
Available online 6 February 2025
1544-6123/

$\mathbb{P}(A)$ denotes the probability of an event A. If the time horizon is short, we assume that $\mathbb{E}(X) = 0$, then $VaR_X(\alpha) = \sigma \cdot VaR_Z(\alpha)$, where $\sigma^2 = \mathbb{E}(X^2)$ is the variance of X, and $Z = X/\sigma$ is the standardized r.v. with distribution D, i.e., $Z \sim D(0, 1)$. The corresponding ES is obtained as $ES_X(\alpha) = \frac{1}{\alpha}\int_0^\alpha VaR_X(u)\, du = \frac{\sigma}{\alpha}\int_0^\alpha VaR_Z(u)\, du = \sigma \cdot ES_Z(\alpha)$.

In this section, we examine several (VaR and ES) TRM bounds. The first subsection revisits two upper bounds commonly used in recent literature, focusing only on the first two moments of X. The second subsection examines a new TRM bound based on the first four moments of X. Notably, the upper tail probability bound under the (two-sided) Chebyshev's (1867) inequality is $\mathbb{P}(X \geq \sigma\xi) \leq \mathbb{P}(|X| \geq \sigma\xi) \leq \xi^{-2}$ for $\xi > 0$. Hence, the implicit VaR upper bound is $VaR_X(\alpha) \leq \frac{\sigma}{\sqrt{\alpha}} \equiv VaR_X^{Ch}(\alpha)$. Integrating the previous inequality, the ES upper bound is $ES_X(\alpha) \leq \frac{2\sigma}{\sqrt{\alpha}} \equiv ES_X^{Ch}(\alpha)$.

2.1. Two-moment TRM bounds

We examine both Cant and OSVP TRM bounds. Cant inequality is defined as $\mathbb{P}(X \geq \sigma\xi) \leq \left(1 + \xi^2\right)^{-1}$. Barrieu and Scandolo (2015) obtain the TRM upper bounds from the previous inequality as $VaR_X(\alpha) \leq \sigma\sqrt{\frac{1-\alpha}{\alpha}} \equiv VaR_X^{Cant}(\alpha)$. The ES upper bound is $ES_X(\alpha) \leq \frac{\sigma}{\alpha}\int_0^\alpha \sqrt{\frac{1-u}{u}}\, du = \frac{\sigma}{\alpha}\left[\arcsin(\sqrt{\alpha}) + \sqrt{\alpha(1-\alpha)}\right] \equiv ES_X^{Cant}(\alpha)$.[1] It holds that the Cant TRM bounds improve the Chebyshev (1867) TRM bounds.

As a refinement, Mercadier and Strobel (2021) proposed a one-sided version of two-sided VP inequality by assuming unimodality for the financial returns' distribution. The OSVP inequality is defined as $\mathbb{P}(X \geq \sigma\xi) \leq \frac{4}{9}\left(1 + \xi^2\right)^{-1}$ for $\xi \geq \sqrt{5/3}$, and the implicit VaR upper bound is obtained as follows:

$$VaR_X(\alpha) \leq \sigma\sqrt{\frac{4}{9\alpha} - 1} \equiv VaR_X^{OSVP}(\alpha) \text{ for } \alpha \leq 1\Big/6. \tag{1}$$

We obtain the ES upper bound by integrating the inequality (1), as follows:

$$ES_X(\alpha) \leq \frac{\sigma}{\alpha}\int_0^\alpha \sqrt{\frac{4}{9u} - 1}\, du$$

$$= \frac{\sigma}{3\alpha}\left[\frac{4}{3}\arcsin\left(\frac{3\sqrt{\alpha}}{2}\right) + \sqrt{\alpha(4-9\alpha)}\right] \equiv ES_X^{OSVP}(\alpha) \tag{2}$$

We refine $VaR_X^{OSVP}(\alpha)$ over $VaR_X^{Cant}(\alpha)$ through the following VaR ratio (VaRR): $VaRR_{Cant}^{OSVP}(\alpha) = \frac{VaR_X^{OSVP}(\alpha)}{VaR_X^{Cant}(\alpha)} = \sqrt{\frac{4-9\alpha}{9-9\alpha}}$. Hence, $VaRR_{Cant}^{OSVP}(\alpha)$ is a decreasing function of α that is less than one. The refinement of $ES_X^{OSVP}(\alpha)$ over $ES_X^{Cant}(\alpha)$ is the ES ratio (ESR): $ESR_{Cant}^{OSVP}(\alpha) = \frac{ES_X^{OSVP}(\alpha)}{ES_X^{Cant}(\alpha)}$. We also verify that ESR is a decreasing function of α that is less than one.

2.2. Four-moment TRM bounds

In what follows, we assume that $\mathbb{E}\left(|X|^4\right) < \infty$, implying that $\mathbb{E}(X^k) < \infty$ for $0 < k \leq 4$. In addition, we denote the respective skewness and kurtosis of X by $\gamma_3 = \mathbb{E}(X^3)/\sigma^3$ and $\gamma_4 = \mathbb{E}(X^4)/\sigma^4$. We propose an upper tail bound for Bhat-VaR given by the following one-sided tail probability bound:

$$\mathbb{P}(X \geq \sigma\xi) \leq \psi(\gamma_3, \gamma_4, \xi), \tag{3}$$

where

$$\psi(\gamma_3, \gamma_4, \xi) = \frac{h(\gamma_3, \gamma_4)}{h(\gamma_3, \gamma_4)\left(1 + \xi^2\right) + g(\gamma_3, \xi)^2}, \tag{4}$$

such that

$$h(\gamma_3, \gamma_4) = \gamma_4 - \gamma_3^2 - 1 > 0 \tag{5}$$

[1] It is observed that this expression does not align with equation (8) in Barrieu and Scandolo (2015).

is the positive definiteness condition of moment matrices,[2] with $\xi > 0$ satisfying the following:

$$g(\gamma_3, \xi) = \xi^2 - \gamma_3\xi - 1 > 0. \tag{6}$$

This condition is verified for $\xi > \xi_g$ with $g(\gamma_3,\xi_g) = 0$. Therefore, ξ_g is a positive root, depending on the skewness, which is obtained as follows:

$$\xi_g(\gamma_3) = (\gamma_3 / 2) + \sqrt{(\gamma_3/2)^2 + 1}. \tag{7}$$

where $\partial\psi(\gamma_3,\gamma_4,\xi)/\partial\gamma_4 > 0$ and $\partial\psi(\gamma_3,\gamma_4,\xi)/\partial\xi < 0$ regardless the sign of γ_3. Nevertheless, $\partial\psi(\gamma_3,\gamma_4,\xi)/\partial\gamma_3 > 0$ when $\gamma_3 > 0$ for $\xi > \xi_g$.

Remark 1. Given equations (5) and (6), the Bhat probability bound of $\psi(\gamma_3,\gamma_4,\xi)$ in (4) can be written as $\left[(1+\xi^2) + \left(\frac{g(\gamma_3,\xi)^2}{h(\gamma_3,\gamma_4)}\right)\right]^{-1}$. Since $h(\gamma_3,\gamma_4) > 0$, then $\psi(\gamma_3,\gamma_4,\xi) < (1 + \xi^2)^{-1}\ \forall\xi$. Therefore, the Bhat probability bound is lower than the Cant bound.

Remark 2. To obtain the Bhat-VaR, consider the equation $q(\gamma_3,\gamma_4,\alpha, \xi) = 0$ with $q(\gamma_3,\gamma_4,\alpha, \xi) = \psi(\gamma_3,\gamma_4,\xi) - \alpha$, such that the following holds:

$$q(\gamma_3, \gamma_4, \alpha, \xi) = \xi^4 - 2\gamma_3\xi^3 + (\gamma_4 - 3)\xi^2 + 2\gamma_3\xi + \theta(\gamma_3, \gamma_4, \alpha), \tag{8}$$

and

$$\theta(\gamma_3, \gamma_4, \alpha) = 1 - \alpha^{-1}(1 - \alpha)h(\gamma_3, \gamma_4). \tag{9}$$

Let $q(\xi)$ denote the shortening of $q(\cdot)$ in equation (8), indicating a polynomial of order four with respect to ξ. Let Δ_q denote the discriminant of the equation $q(\xi) = 0$.

Proposition 1. *Consider the condition in equation* (5), $\gamma_4 > (1 - \alpha)^{-1}$ *and* $q(\xi) = 0$ *with* $q(\xi)$ *in equation* (8). *If the coefficient of* $\theta(\gamma_3,\gamma_4,\alpha)$ *in equation* (9) *is negative and* $\Delta_q < 0$, *then the equation has only one positive root* ξ_ψ, *which depends on* $(\gamma_3,\gamma_4,\alpha)$, i.e., $\xi_\psi = \xi_\psi(\gamma_3,\gamma_4,\alpha)$. *Therefore, the Bhat-VaR upper bound of X at confidence level* α *is as follows:*

$$VaR_X(\alpha) \le \sigma \cdot \xi_\psi(\gamma_3, \gamma_4, \alpha) \equiv VaR_X^{Bhat}(\alpha). \tag{10}$$

Remark 3. In Proposition 1, we do not consider the case $\theta(\gamma_3,\gamma_4,\alpha) > 0$ in order to simplify the results, especially since it corresponds to extremely small kurtosis values (i.e., $1 < \gamma_4 < (1 - \alpha)^{-1}$ for $\alpha \le 0.05$). Note also that we use losses from daily stock returns, which are generally leptokurtic.

Corollary 1. *The closed-form expression for* $VaR_Z^{Bhat}(\alpha)$ *in Proposition 1 with* $\gamma_3 = 0$ *is obtained as follows:*

$$\xi_\psi(0, \gamma_4, \alpha) = \sqrt{-\frac{(\gamma_4 - 3)}{2} + \sqrt{\frac{(\gamma_4 - 3)^2}{4} - \alpha^{-1}(1 - (1 - \alpha)\gamma_4)}}. \tag{11}$$

The subsequent result demonstrates that Bhat-VaR is tighter than OSVP-VaR in most cases under unimodal and symmetric distributions for losses, leading to the following corollary:

Corollary 2. *Equations* (1) *and* (11) *verify that* $VaRR_{OSVP}^{Bhat}(\alpha) = \frac{\xi_\psi(0,\gamma_4,\alpha)}{\sqrt{\frac{4}{9\alpha}-1}} \le 1$, *if and only if* $\underline{\omega}(\alpha) \le \gamma_4 \le \overline{\omega}(\alpha)$ *for* $\alpha \le 0.1580$ *where* $\overline{\omega}(\alpha) = \frac{1}{5}\left(-11 + 36\alpha + \frac{16}{9}\alpha^{-1}\right)$ *and* $\underline{\omega}(\alpha) = (1 - \alpha)^{-1}$.

Next, considering equation (10), the Bhat-ES upper bound is obtained as follows:

$$ES_X(\alpha) \le \frac{\sigma}{\alpha}\int_0^\alpha \xi_\psi(\gamma_3, \gamma_4, u)\, du \equiv ES_X^{Bhat}(\alpha), \tag{12}$$

Similar to Kratz et al. (2018), it is obtained using the following approximation:

$$ES_X^{Bhat}(\alpha) \approx \frac{1}{n}\sum_{j=1}^{n} VaR_X^{Bhat}\left(\frac{j\alpha}{n}\right) = \frac{\sigma}{n}\sum_{j=1}^{n} \xi_\psi\left(\gamma_3, \gamma_4, \frac{j\alpha}{n}\right). \tag{13}$$

[2] See Widder (1946), noting that this condition implies that $\gamma_4 > 1$.

At this time, we set $n = 1,000$.[3] The following VaR and ES ratios measure tightness between OSVP-VaR (ES) and Bhat-VaR (ES) upper bounds:

$$VaRR_{OSVP}^{Bhat}(\alpha) = \frac{VaR_X^{Bhat}(\alpha)}{VaR_X^{OSVP}(\alpha)};\ ESR_{OSVP}^{Bhat}(\alpha) = \frac{ES_X^{Bhat}(\alpha)}{ES_X^{OSVP}(\alpha)}. \quad (14)$$

Fig. 1 presents four panels, denoted as A, B, C, and D. Panel A exhibits the plot of the ratio $ES_X^{OSVP}(\alpha)/VaR_X^{OSVP}(\alpha)$ as a function of $\alpha \in (0, 0.05]$. This ratio increases with α such that $ES_X^{OSVP}(\alpha) \approx 2 \cdot VaR_X^{OSVP}(\alpha)$. Panel B presents three plots of $ES_X^{Bhat}(\alpha)/VaR_X^{Bhat}(\alpha)$ being increasing with α, where each exhibits a different kurtosis level, $\gamma_4 = 5, 10, 20$, but the same skewness level $\gamma_3 = 0.25$. Note that a higher γ_4 indicates a higher ratio. The range of these ratios is (1.32, 1.46). Panels C and D respectively exhibit the plots of $VaRR_{OSVP}^{Bhat}(\alpha)$ and $ESR_{OSVP}^{Bhat}(\alpha)$ in equation (14) as functions of α for the same values of γ_3 and γ_4 in panel B. The finding reveals that $VaRR_{OSVP}^{Bhat}(\alpha)$ increases as γ_4 or α increases, indicating that; for instance, the Bhat-VaR is tighter than the OSVP-VaR when α is slightly lower than 0.015 with a high kurtosis of 20 since the graph of this ratio is below the horizontal line of value one. Note that for regulatory VaR ($\alpha = 0.01$),[4] $VaRR_{OSVP}^{Bhat}(0.01)$ is lower than one for all previous levels of kurtosis. Furthermore (but not reported), $VaRR_{OSVP}^{Bhat}(\alpha)$ also increases with higher skewness levels. Finally, $ESR_{OSVP}^{Bhat}(\alpha)$ increases as γ_4 or α increases, and it is always lower than one; hence, ES-Bhat is tighter than ES-OSVP for reasonable kurtosis values of daily financial returns series.

The Basel multiplier, which is crucial for setting the upper bounds for VaR scenarios, is a VaR ratio defined as $m^i(\alpha) \equiv VaRR_N^i(\alpha) = \frac{VaR_X^i(\alpha)}{VaR_X^N(\alpha)}$ where $i =$ OSVP, Bhat, and $VaR_X^N(\alpha) = \sigma \cdot \Phi^{-1}(1-\alpha)$, with $\Phi(\cdot)$ as the standard Normal (N) cdf. For example, if $\alpha = 0.01$ the denominator becomes $\Phi^{-1}(0.99) = 2.3263$. Then, $m^{OSVP}(0.01) = 2.8333$ as Mercadier and Strobel (2021) proposed, in contrast to a tighter multiplier $m^{Bhat}(0.01) = 1.6104$, indicating lesser capital requirements.

3. Performance of TRM bounds

In this section, we examine the performance of OSVP- and Bhat-TRM bounds for daily losses, which are represented as X_t. First, we assume that daily losses X_t follow Hansen's (1994) skewed-t (SKT) distribution. Next, we employ Glosten et al.'s (1993) GJR model for daily stock returns.

3.1. Assuming a skewed-t distribution

Suppose that daily losses are given by $X_t = Z \sim D(0,1)$, with a cumulative distribution function (cdf) denoted as $F_D(z;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the parameter set. Assume that Z follows the SKT distribution with parameters $\lambda \in (-1,1)$ and $\upsilon > 2$, i.e., $Z \sim SKT(\upsilon,\lambda)$. Both $\gamma_3^{SKT} = \mathbb{E}(Z^3)$ and $\gamma_4^{SKT} = \mathbb{E}(Z^4)$ depend on the parameter set (υ,λ), i.e., $\gamma_j^{SKT} = \gamma_j^{SKT}(\upsilon,\lambda)$ with $j = 3, 4$, with expressions that mirror Eqs. (2) and (3) in Jondeau and Rockinger (2003). We next compare $VaR_Z^{Bhat}(\alpha)$ and $VaR_Z^{OSVP}(\alpha)$ performance concerning the true (upper tail) VaR, or VaR under the SKT distribution (SKT-VaR hereafter), which is obtained from the inverse of the SKT cdf, i.e., $VaR_Z^{SKT}(\alpha) = F_{SKT}^{-1}(1-\alpha;\upsilon,\lambda)$, for the particular case of $\alpha = 0.01$ and different values of (υ,λ).[5] We rerun the same previous analysis considering the respective ES measures. In short, this study is based on the following VaR and ES ratios:

$$VaRR_{SKT}^i(0.01) = \frac{VaR_Z^i(0.01)}{VaR_Z^{SKT}(0.01)};\ ESR_{SKT}^i(0.01) = \frac{ES_Z^i(0.01)}{ES_Z^{SKT}(0.01)}, \quad (15)$$

where $i = OSVP,\ Bhat$. Note that $VaR_Z^{Bhat}(0.01) = \xi_\psi(\gamma_3^{SKT}, \gamma_4^{SKT}, 0.01)$ is computed according to Proposition 1, and $ES_Z^{Bhat}(0.01)$ is obtained given equation (13), $VaR_Z^{OSVP}(0.01) = 6.59$, $ES_Z^{OSVP}(0.01) = 13.28$ and $ES_Z^{SKT}(0.01) \approx \frac{1}{n}\sum_{j=1}^n VaR_Z^{SKT}\left(\frac{0.01\ j}{n}\right)$ with $n = 1,000$.

Fig. 2 presents panels A and B for VaRR, while panels C and D for ESR according to equation (15). The curved line in each panel is constructed using the skewness and kurtosis levels from the SKT distribution for a range of typical values of $\upsilon \in [4.4, 7]$ and $\lambda = 0.02$, 0.06, 0.12, i.e., $(\gamma_3^{SKT}(\upsilon,\lambda), \gamma_4^{SKT}(\upsilon,\lambda))$ where λ is fixed along each curve. In addition, the values of γ_3 and γ_4 to compute the $VaR_Z^{Bhat}(0.01)$ are obtained from the same values of υ and λ used to calculate $VaR_Z^{SKT}(0.01)$. The results reveal that the highest values for kurtosis (skewness) in each curve are 18.04 (0.11) for $\lambda = 0.02$, 18.37(0.33) for $\lambda = 0.06$ and 19.44 (0.66) for $\lambda = 0.12$.[6]

Five notable findings emerge from Fig. 2. First, the curves in panel A (related to $VaRR_{SKT}^{OSVP}(0.01)$) increase since the denominator $VaR_Z^{SKT}(0.01)$ decreases when increasing υ or λ while the numerator remains constant. Second, in contrast to panel A, the curves in

[3] The finite sum for computing ES in equation (13) is nested into a broader framework established by Leorato et al. (2012), which is expressed as $ES_X(\alpha) = \sum_{j=1}^n w_j VaR_X(\alpha_j)$ where $VaR_X(\alpha_j)$ represents the α_j-quantile of losses, with $\alpha_j \le \alpha$, and w_j denotes the corresponding weight, ensuring that $\sum_{j=1}^n w_j = 1$. Exploring alternative structures for the w_j weights is encouraged for future research.

[4] This study focuses on the regulatory VaR since it is recommended by the Basel Committee on Banking Supervision (2019).

[5] We use Kevin Shephard's MATLAB function *skewtinv* to compute $F_{SKT}^{-1}(0.99;\upsilon,\lambda)$.

[6] The range of $\upsilon \ge 4.4$ yields kurtosis values that are consistent with empirical evidence for daily returns. The same applies to λ for the skewness. Furthermore, other ranges for both parameters have been considered without altering our main conclusions.

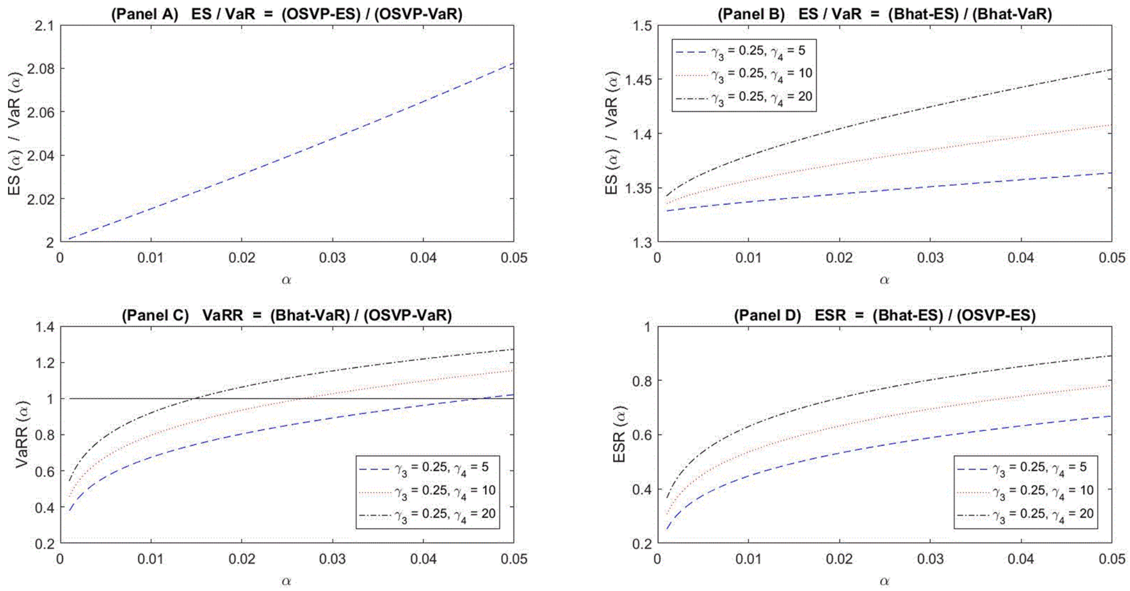

Fig. 1. ES/VaR, VaRR, and ESR for OSVP and Bhat.
Notes: This figure presents plots of ES versus VaR for OSVP inequality (Panel A), Bhat inequality (Panel B), VaR upper bound ratios (VaRR) for Bhat-VaR vs OSVP-VaR (Panel C), and ES ratios (ESR) for Bhat-ES vs OSVP-ES (Panel A). All panels exhibit curved lines for kurtosis values = 5, 10, 20 and fixed skewness values = 0.25, as a function of α∈(0,0.05].

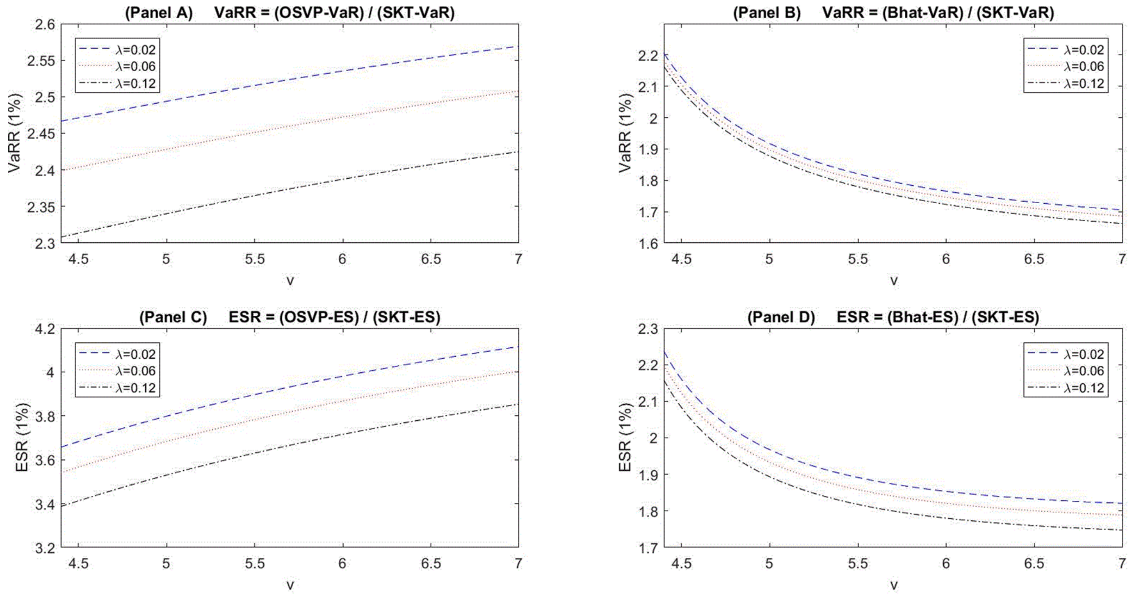

Fig. 2. One percent Bhat-VaRR (ESR) and OSVP-VaRR (ESR) for SKT distribution.
Notes: This figure plots one percent VaR ratios (VaRR) and ES ratios (ESR) for upper bound OSVP-VaR over SKT-VaR (Panel A), upper bound Bhat-VaR over SKT-VaR (Panel B), upper bound OSVP-ES over SKT-ES (Panel C) and upper bound Bhat-ES over SKT-ES (Panel D). Each curved line in each graph has been constructed with a range of typical values of υ ∈[4.4,7] and λ=0.02,0.06,0.12, such that λ is fixed along each curved line.

panel B (related to $VaRR^{Bhat}_{SKT}(0.01)$) indicate that the numerator of VaRR varies at each point. For $\lambda = 0.02$, the $VaRR^{Bhat}_{SKT}(0.01)$ values in panel B range from 2.21 to 1.71 for $\upsilon = 4.4$ to $\upsilon = 7$, while the $VaRR^{OSVP}_{SKT}(0.01)$ values in panel A range from 2.47 to 2.57 for $\upsilon = 4.4$ to $\upsilon = 7$. For $\lambda = 0.12$, the $VaRR^{Bhat}_{SKT}(0.01)$ values in panel B range from 2.16 to 1.66 for $\upsilon = 4.4$ to $\upsilon = 7$, while the $VaRR^{OSVP}_{SKT}(0.01)$ values in panel A range from 2.31 to 2.42 for $\upsilon = 4.4$ to $\upsilon = 7$. Third, $VaRR^{Bhat}_{SKT}$ is smaller than $VaRR^{OSVP}_{SKT}$ for all values of (υ,λ). To further examine this result, we study the behavior of $VaRR^{Bhat}_{OSVP}(0.01)$ using equation (14), which is straightforwardly inferred from previous VaR ratios. We verify, but not exhibit here to save space, that the $VaRR^{Bhat}_{OSVP}(0.01)$ values are always lower than one (in line with the findings in panel C in Fig. 1 when $\alpha = 0.01$). In summary, Bhat-VaR is tighter than OSVP-VaR for the different values of (υ, λ) used in Fig. 2. Fourth, the curves in panel A increase whilst those in panel B decrease with υ. Fifth, the behavior of curves in panel C (panel B) for the ES ratios closely resemble those in panel A (panel D) for OSVP (Bhat).

3.2. GJR-GARCH model

Assume that daily losses are given by $X_t = -R_t$, where R_t is the r.v. for stock returns. This follows a conditionally heteroskedastic process that is described by $R_t = \mu_t + \varepsilon_t$, with $\varepsilon_t = \sigma_t Z_t$ where μ_t and σ_t^2 respectively denote the conditional mean and variance of X_t given the information set $\mathscr{F}_{t-1}$. Additionally, $\{Z_t\}$ is a sequence of independent and identically distributed (iid) r.v. with zero mean and unit variance, i.e., $E(Z_t) = 0$ and $E(Z_t^2) = 1$.

We model the error process $\{\varepsilon_t\}$ as a conditionally heteroskedastic white noise sequence, $\sigma_t^2 = E(\varepsilon_t^2 | \mathscr{F}_{t-1})$, such that $\sigma_t^2 = \alpha_0 + c(Z_{t-1})\sigma_{t-1}^2$, where $\alpha_0 > 0$, $c(Z_t)$ is a well-defined function verifying that $\sigma_t^2 > 0$ for all t. He and Teräsvirta (1999) and Ling and McAleer (2002) examined the moment properties of alternative GARCH-family models. We use the GJR model, where $c(Z_t) = \beta + \alpha^+ (Z_t^+)^2 + \alpha^- (Z_t^-)^2$, with $\beta \geq 0$, $\alpha^+ \geq 0$ and $\alpha^- \geq 0$. In this context, we define $Z_t^+ = \max(Z_t, 0)$ and $Z_t^- = \min(Z_t, 0)$.

If $E|Z_t^{2k}| < \infty$, then the necessary and sufficient condition for the $2k$-th moment of ε_t is $a_k = E[c(Z_t)^k] < 1$. Assuming $\mu_t = 0$ for daily losses, then $X_t = -\varepsilon_t$. The GJR unconditional variance, skewness, and kurtosis of X_t are respectively denoted as $\sigma_{GJR}^2 \equiv E(\varepsilon_t^2)$, $\gamma_3^{GJR} \equiv -E(\varepsilon_t^3)/\left(E(\varepsilon_t^2)\right)^{3/2}$, and $\gamma_4^{GJR} \equiv E(\varepsilon_t^4)/\left(E(\varepsilon_t^2)\right)^2$, which are the expressions: $\sigma_{GJR}^2 = E(\sigma_t^2) = \alpha_0(1-a_1)^{-1}$, $\gamma_3^{GJR} = -E(Z_t^3)E\left[(\sigma_t^2)^{3/2}\right] / \left(E(\sigma_t^2)\right)^{3/2}$, and $\gamma_4^{GJR} = E(Z_t^4)E(\sigma_t^4)/\left(E(\sigma_t^2)\right)^2 = E(Z_t^4)\left(1-a_1^2\right)(1-a_2)^{-1}$, such that $E(\sigma_t^{2k}) < \infty$ if $a_k < 1$ for $k = 1, 2$. It is verified that $a_1 = \beta + \alpha^+ + (\alpha^- - \alpha^+)E\left[(Z_t^-)^2\right]$ and $a_2 = \beta^2 + 2\beta\alpha^+ + (\alpha^+)^2 E(Z_t^4) + \left[(\alpha^-)^2 - (\alpha^+)^2\right]E\left[(Z_t^-)^4\right] + 2\beta(\alpha^- - \alpha^+)E\left[(Z_t^-)^2\right]$; see León and Ñíguez (2020). If we assume that Z_t follows a symmetric distribution, for instance, $Z_t \sim N(0,1)$ then $E(Z_t^3) = 0$, $E(Z_t^4) = 3$, $E\left[(Z_t^-)^2\right] = 1/2$ and $E\left[(Z_t^-)^4\right] = 3/2$. The unconditional skewness and kurtosis for losses under the GJR model with Normal innovations (henceforth, GJR-N) are $\gamma_3^{GJR-N} = 0$ and

$$\gamma_4^{GJR-N} = 3\left(\frac{1-\beta^2-\beta(\alpha^+ + \alpha^-) - \frac{1}{4}(\alpha^+ + \alpha^-)^2}{1-\beta^2-\beta(\alpha^+ + \alpha^-) - \frac{3}{2}\left[(\alpha^+)^2 + (\alpha^-)^2\right]}\right). \tag{16}$$

By plugging γ_4^{GJR-N} in equation (16) into equation (11) in Corollary 1, we finally obtain the corresponding $VaR_Z^{Bhat}(\alpha) \equiv \xi_\psi\left(0, \gamma_4^{GJR-N}, \alpha\right)$ for the standardized losses. Therefore, $VaR_X^{Bhat}(\alpha) = \sqrt{\sigma_{GJR-N}^2} \cdot \xi_\psi\left(0, \gamma_4^{GJR-N}, \alpha\right)$ with $\sigma_{GJR-N}^2 = \alpha_0\left[1 - \beta - \frac{1}{2}(\alpha^- + \alpha^+)\right]^{-1}$.

If we assume that $Z_t \sim SKT(\upsilon, \lambda)$, then $E(Z_t^3) = \gamma_3^{SKT}(\upsilon, \lambda)$ and $E(Z_t^4) = \gamma_4^{SKT}(\upsilon, \lambda)$ respectively represent Eqs. (2) and (3) in Jondeau and Rockinger (2003). Concerning $E\left[(Z_t^-)^2\right]$ and $E\left[(Z_t^-)^4\right]$, which we compute numerically to calculate a_1 and a_2 under the SKT distribution. Then, the GJR unconditional variance and kurtosis under SKT distributed innovations (henceforth, GJR-SKT), denoted as $\sigma_{GJR-SKT}^2$ and $\gamma_4^{GJR-SKT}$, are easily obtained. Nevertheless, the unconditional skewness $\gamma_3^{GJR-SKT}$ is calculated by an approximation of the numerator, $E\left[(\sigma_t^2)^{3/2}\right]$, using a second-order Taylor series expansion, $E\left[(\sigma_t^2)^{3/2}\right] \approx \frac{5}{8}\left(E(\sigma_t^2)\right)^{3/2} + \frac{3}{8}E(\sigma_t^4)\left(E(\sigma_t^2)\right)^{-1/2}$; see Alexander et al. (2021). Finally, we obtain $VaR_X^{Bhat}(\alpha)$ under the GJR-SKT model according to Proposition 1.

4. Empirical results

We present our theoretical results on OSVP and Bhat-VaR bounds for losses through an empirical exercise using daily log returns calculated as $r_t = \ln(p_t/p_{t-1})$ from daily closing prices p_t for S&P100 constituents, sampled from June 8, 2017, to March 8, 2023, with a total of $T = 1,500$ observations from DataStream. Daily losses are defined as $x_t = -r_t$.

We calculate VaR upper bounds at the 1 % level with a rolling window of 1, 000 days, starting from an in-sample period of *T-N* observations followed by a $N = 500$ days out-of-sample (OOS) period from April 8, 2021, to March 8, 2023. The rolling-window procedure involves computing VaR for the initial 1, 000-day window, then shifting the window forward by one day and recalculating until the entire OOS period is covered.

The ranges of all daily sample standard deviation, kurtosis, and skewness series across all stock losses are [0.0117, 0.0426], [5.54, 60.5], and [− 1.10, 3.52], respectively. The sample means for all previous daily series of standard deviation, kurtosis, and skewness are 0.0205, 15.1327 and 0.371, respectively.

We initially consider VaR upper bound estimates obtained by plugging the sample standard deviation, skewness, and kurtosis into equations (1) and (10),[7] obtaining daily series for $VaRR_{OSVP}^{Bhat}(0.01)$ estimates given in equation (14) for each stock over the OOS period. We find that $VaRR_{OSVP}^{Bhat}(0.01) < 1$ for 90 out of 100 stocks. Among the other 10 stocks, $VaRR_{OSVP}^{Bhat}(0.01) > 1$ every day for 4 of them but not for the remaining 6 stocks. These results provide evidence that Bhat-VaR is sharper than OSVP-VaR.

[7] Since the sample means of daily losses are approximately zero, they are excluded from the calculation of the VaR upper bounds.

We next examine the number of hits (i.e., the number of times losses are higher than the actual VaR upper bound). Let $w^i_{k,t}(\alpha) = \mathscr{I}\left(x_{k,t} > VaR^i_{X_{k,t}}(\alpha)\right)$ denote the violation or hit variable, where $i = OSVP, Bhat$, and $x_{k,t}$ is an observation of the loss r.v. $X_{k,t}$ for stock $k = 1, \ldots, 100$ on day t of the OOS period. Also, let $\widehat{w}^i_{k,t}(\alpha) = \mathscr{I}\left(x_{k,t} > \widehat{VaR}^i_{X_{k,t}}(\alpha)\right)$ be the estimate of $w^i_{k,t}(\alpha)$. For the Bhat-VaR: $\widehat{w}^{Bhat}_{k,t}(\alpha) = \mathscr{I}\left(x_{k,t} > \widehat{VaR}^{Bhat}_{X_{k,t}}(\alpha)\right)$ where $\widehat{VaR}^{Bhat}_{X_{k,t}}(\alpha) = \widehat{\sigma}^{(k)}_{t|t-1} \cdot \xi_\psi\left(\widehat{\gamma}^{(k)}_{3,t|t-1}, \widehat{\gamma}^{(k)}_{4,t|t-1}, \alpha\right)$ represents the Bhat-VaR forecast for losses obtained by plugging into (10) the sample standard deviation, $\widehat{\sigma}^{(k)}_{t|t-1}$, skewness, $\widehat{\gamma}^{(k)}_{3,t|t-1}$, and kurtosis, $\widehat{\gamma}^{(k)}_{4,t|t-1}$ given the information set at day $t-1$ determined by the rolling-window procedure for the one-day-ahead forecast of $\sigma^{(k)}$, $\gamma^{(k)}_3$, and $\gamma^{(k)}_4$, respectively. Let $\widehat{W}^i_k(\alpha) = \sum_{t=1}^{500} \widehat{w}^i_{k,t}(\alpha)$ be the estimate of the total number of daily violations of stock k through the OOS period (500 days), then $\widehat{W}^i(\alpha) = \sum_{k=1}^{100} \widehat{W}^i_k(\alpha)$ is the estimate of the number of violations across all stocks. As expected from our theoretical findings, our results demonstrate that Bhat-VaR is smaller than OSVP-VaR since $\widehat{W}^{OSVP}(0.01) = 17$ and $\widehat{W}^{Bhat}(0.01) = 27$.[8]

Finally, we estimate GJR-N across all daily loss series using the maximum likelihood model, each of which includes 1, 500 observations. We find that 87 of the 100 stocks have finite second and fourth moments, i.e., $a_1 < 1$ and $a_2 < 1$. Additionally, the estimated kurtosis obtained using equation (16) for these stocks is lower than 20. Moreover, the estimated $VaRR^{Bhat}_{OSVP}(0.01)$ is below 1 for all 87 stocks, indicating that the Bhat-VaR bounds in the previous model are tighter compared with OSVP estimates.

5. Conclusions

This study establishes upper bounds for TRM, such as VaR and ES, using Bhattacharyya's inequality to incorporate higher-order moments of loss distributions. We demonstrate that the Bhat-VaR (ES) upper bound yields more accurate estimates compared to OSVP inequalities, particularly for reasonable kurtosis levels in daily financial returns, validated through empirical analysis of S&P 100 index stocks.

CRediT authorship contribution statement

M. Ángeles Carnero: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Ángel León:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Trino-Manuel Ñíguez:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of Competing Interest

The authors declare no conflicts of interest.

Acknowledgements

This work was supported by the Spanish Ministry of Science and Innovation [PID2021–124860NB-I00] and Generalitat Valenciana [CIPROM/2021/060].

Appendix

Proof of Proposition 1. Observe that $\theta(\gamma_3,\gamma_4,\alpha) < 0$ if and only if $\gamma_4 - \gamma_3^2 > (1-\alpha)^{-1}$. Moreover, if $\gamma_3 > 0$, then according to Descartes' rule of signs (DRS) ,[9] the polynomial equation $q(\xi) = 0$ will have either one or three positive real roots (and one negative root). Note that this behavior is satisfied regardless of the sign of the coefficient of ξ^2 in $q(\xi)$. If $\Delta_q < 0$, then we can obtain two complex conjugate non-real roots and two distinct real roots.[10] Given the previous DRS analysis, two real roots of different signs are obtained. If $\gamma_3 < 0$, then the DRS analysis suggests that there can be one positive root (with one or three negative roots). Since $\Delta_q < 0$, we also

[8] Tighter Bhat-VaR bounds, which yield more exceptions than OSVP-VaR, can be advantageous in stable market conditions; however, less stringent VaR bounds may be more beneficial during financial crises. Ultimately, Bhat-VaR bounds reflect the stylized features of return distributions more accurately, resulting in a more appropriate minimum capital requirements.

[9] For instance, see Sydsaeter (1981).

[10] Given the general quartic equation $\sum_{i=0}^{4} a_i x^i = 0$, with real coefficients and $a_4 \neq 0$, the nature of its roots is determined by the sign of its discriminant, $\Delta = \left(4\Delta_0^3 - \Delta_1^2\right)/27$, where $\Delta_0 = a_2^2 - 3a_3a_1 + 12a_4a_0$ and $\Delta_1 = 2a_2^3 - 9a_3a_2a_1 + 27a_3^2a_0 + 27a_4a_1^2 - 72a_4a_2a_0$. If $\Delta > 0$, the equation has four distinct roots that can all be real or all imaginaries. If $\Delta < 0$, the equation has two distinct real roots and two complex conjugate non-real roots. If $\Delta = 0$, the equation has at least two equal roots and requires a deeper analysis. For example, see Rees (1922) for more details.

obtain a negative and positive root. According to DRS, the case of $\gamma_3 = 0$ results in two different real roots of different signs. Additionally, since $\Delta_q < 0$, we obtain two roots of different signs. In summary, $VaR_Z^{Bhat}(\alpha)$ is the positive root for each of the previous cases with respect to the sign of γ_3. ■

Proof of Corollary 1**.** The positive root ξ_ψ is easily obtained from $q(0, \gamma_4, \alpha, \xi) = 0$ with $q(\cdot)$ in equation (8).■

Proof of Corollary 2. Let $\alpha^* = 0.1580$ and consider the equation $\overline{\omega}(\alpha) = \underline{\omega}(\alpha)$, which can be rewritten as $-36\alpha^3 + 47\alpha^2 - \frac{160}{9}\alpha + \frac{16}{9} = 0$. This third-degree polynomial has three positive roots with α^* as the only root verifying the OSVP restriction ($\alpha \leq 1/6$). It is verified that $\frac{d\overline{\omega}(\alpha)}{d\alpha} < 0$ and $\frac{d\underline{\omega}(\alpha)}{d\alpha} > 0$ when $\alpha \in (0, \alpha^*)$. Thus, $VaRR_{OSVP}^{Bhat} \leq 1$ when $\alpha \leq \alpha^*$, and $VaRR_{OSVP}^{Bhat} > 1$ for $\alpha \in (\alpha^*, 1/6]$.■

Data availability

The data used is publicly available from DataStream.

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