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Empirical Analysis of the US Swap Curve

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Abstract: This paper provides an empirical analysis of the US swap rate curve using principal components analysis (PCA) to identify the factors which explain the variation in the data. We also investigate the forecasting performance of different econometric models for individual maturities across the curve using daily data over the period 1998 to 2011. The PCA analysis indicates that the first two factors explain approximately 99.76% of the cumulative variation in the sample. We also find that a continuous time modelling approach has a satisfactory performance across the curve based on the RMSE.

Keywords: Continuous Time, Discrete Time, PCA.

I. Introduction

The application of stochastic differential equation models in economics and finance has a number of advantages compared to discrete time models and are outlined in Bergstrom and Nowman (2007) recently. In this paper we investigate the principal component analysis of the US swap rate curve over the period 1998 to 2011 to identify the factors which explain the curve. We also compare the forecasting performance of discrete time econometric models with a model formulated in continuous time for the different maturities across the curve. We find that the continuous time model has a superior forecasting performance across the curve. The rest of the paper is organized as follows: Section 2 outlines the modelling of the swap curve. Section 3 presents the data and empirical results are given in Section 4. Conclusions are presented in Section 5.

2. Modelling the Swap Curve

The model we use for the dynamics of the different swap rate maturities was developed by Chan, Karolyi, Longstaff and Sanders (1992, CKLS).

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$$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{\gamma}(t)\zeta \qquad (dt(t \geq 0)) \tag{1}$$

where $\{r(t), t > 0\}$ is a swap rate maturity, α and β are the unknown drift and mean reversion structural parameters; σ is the volatility of the rate; γ is the proportional volatility exponent and $\zeta(dt)$ is a white noise error term. The parameters are estimated using a discrete model in Nowman (1997). We also estimate well known ARMA, ARIMA and Autoregressive Fractionally Integrated Moving Average (ARFIMA) models. We begin with the *ARMA(p, q),* where this model will have *p* autoregressive and *q* moving average terms. The ARMA model is therefore specified as follows:

$$\phi(L) = \mu + \varepsilon_t - \theta(L)\varepsilon_t \tag{2}$$

where $\phi(L)$ and $\theta(L)$ denote the polynomials in the lag operator; hence $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p$ and $\theta(L) = 1 - \theta_1 L - \vartheta_2 L^2 - ... - \theta_q L^q$. One of the underlying assumptions for the ARMA models is that the underlying data series follows a stationary, i.e., *I(0),* process; therefore, should one apply the ARMA model to a non-stationary data series, the results would be spurious.

The discrete time analysis continues with the *ARFIMA(p,d,q* model, developed by Box and Jenkins (1976), which provides a contrast to the ARMA model by assuming that the underlying data series follows a non-stationary process. Once again this has *p* autoregressive and q moving average terms, as was the case of the ARMA model; however, this model extends the ARMA model in that it also has a *d* component, where this measures the number of times that the underlying data series has to be differenced in order to make the process stationary, where $d \geq 1$ and an integer. The ARIMA model is therefore specified as:

$$\phi(L)\left[(1-L)^d y_t\right] = \mu + \theta(L)\varepsilon_t \tag{3}$$

where $\phi(L)$ and $\theta(L)$ denote the polynomials in the lag operator, and $(1-L)^d = \Delta^d y_t$ is the *d*th difference of y_t. The final alternative model is the *ARFIMA(p,d,q),* first introduced by Granger and Joyuex (1980), Granger (1980, 1981) and Hosking (1981), where the assumption is made that the underlying data series follow a mean reverting process, however, the Wold decomposition and the autocorrelation coefficients for this process will exhibit a very slow hyperbolic rate of decay, where, the higher the value of *d*, the slower the rate of decay. Like the ARIMA model, it also has a *d* component, however, in this case $0<d<1$. The ARFIMA model parameterises the conditional mean of the series generating process as an ARFIMA (p ,d, q) process, which is specified as follows:

$$\phi(L)(1-L)^d\left(y_t - \mu\right) = \theta(L)\varepsilon_t \tag{4}$$

where $\phi(L)$ and $\theta(L)$ denote the polynomials in the lag operator, where all the roots of $\phi(L)$ and $\theta(L)$ lie outside the unit root circle; d denotes the fractional differencing parameter; and ε_t is white noise. This model is estimated using the Maximum Likelihood Estimation (MLE) method outlined in Sowell (1986, 1992).

3. Data

The dataset used in the empirical work consists of daily US swap rates obtained from Datastream for the 1, 2, 4, 10, 15, 20, 25 and 30 years rates. The rates are sampled from June 1998 to December 2011. There is a total of 3566 observation dates and at each date there are *N*-interest rates *(N=8)*. Table 1 reports the summary statistics and Figure 1 displays the swap curve evolutions over the period. The mean of the data varies from 3.2719 percent for the 1-year rate to 5.2982 percent for the 30-year rate with standard deviations of 2.1036 percent and 1.0972 percent. The ADF statistics do not reject the null hypothesis of a unit root in the level series.

Table 1: Descriptive Statistics

	1-Year	2-Year	4-Year	10-Year	15-Year	20-Year	25-Year	30-Year
$r(t)$								
Mean	3.2719	3.5687	4.0976	4.8739	5.1419	5.2498	5.2834	5.2982
SD	2.1036	1.9488	1.6647	1.2599	1.1603	1.1284	1.1090	1.0972
ADF	-0.8962	-1.2305	-1.7430	-2.6281	-2.9100	-2.9882	-2.9833	-2.9777
$\Delta r(t)$								
Mean	-0.0015	-0.0015	-0.0014	-0.0012	-0.0011	-0.0011	-0.0010	-0.0010
SD	0.0446	0.0581	0.0654	0.0663	0.0637	0.0619	0.0608	0.0601
ADF	-54.925	-58.012	-59.102	-59.709	-59.931	-59.916	-59.979	-59.651

Note: Mean, standard deviations of daily swap rates. The variable $r(t)$ is the level and $\Delta r(t)$ is the daily change. ADF denotes the Augmented Dickey-Fuller unit root statistic.

Using the swap rate data described above, we also perform a principal components analysis (PCA) on the sample covariance matrix to identify the factors which explain variation in the data. This transforms original dataset into variables that maximize the explained variance of the group and are as uncorrelated as possible (i.e. each variable is orthogonal to one another). Since the variables are orthogonal, each factor is uniquely determined, up to a sign change.

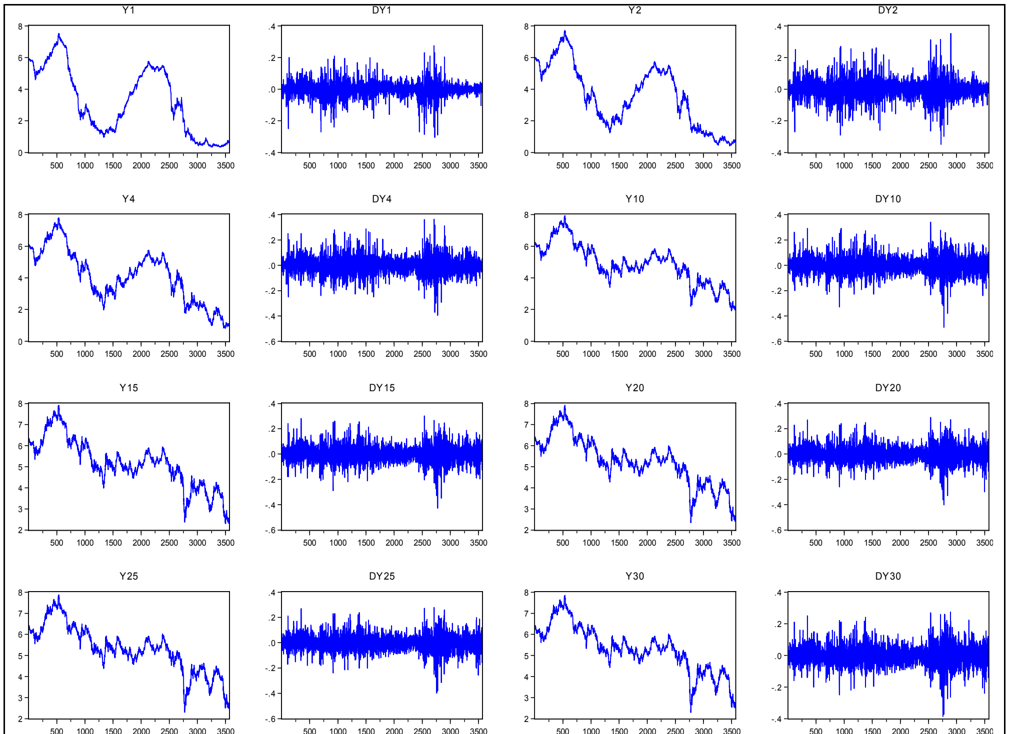

Figure 1: US Swap Rate Curves

PCA starts from the assumption that the covariance matrix for the data, Σ, can be decomposed into $\Gamma\Lambda\Gamma^{T}$, where Γ is an $N \times N$ orthogonal matrix containing factor loadings and Λ is an $N \times N$ diagonal matrix containing ***N*** eigenvalues, N being the number of swap rates. Denoting our original dataset by ***X***, each subsequent variable is defined to be $\Gamma' X$. As the variance of each factor is given by its corresponding eigenvalue, each variable is ordered based upon the size of its eigenvalue (Flury (1988) for more details).[1] The variable with the largest eigenvalue is the first principal component, while the variable with the second largest eigenvalue is the second principal component, and so on. As they are mathematical constructs, principal component factors are latent or unobservable in nature. The simplest way to interpret factors is to examine the effects of a shock to the factor on swap rates. To accomplish this task, we plot the factor loading coefficients and provide a description of their shape.

[1] To see this we denote each variable or factor as, V. Since $V = \Lambda' X$, var(V) = var($(\Gamma' X) = \Gamma' \operatorname{var}(X)\Gamma$. Since var(X)=Σ, var($(\Gamma' X) = \Gamma' \Sigma \Gamma = \Lambda$ owing to the orthogonality of the Γ matrix. Here, Λ is an NxN matrix containing the eigenvalues of the sample covariance matrix of the group.

We use principal components analysis to estimate factor loadings, which are displayed in Table 2, and plot the coefficients for the first three factors in Figure 2. Factor loadings also correspond to ordinary least squares (OLS) regression coefficients which would result from an OLS regression of swap rates on factors. Each principal component coefficient measures the relative change in the swap rate to a shock in the corresponding factor.

Table 2: Factor Loadings

	Factor 1	Factor 2	Factor 3		Factor 1	Factor 2	Factor 3
1-Year	0.493	-0.570	0.513	**15-Year**	0.275	0.324	-0.004
2-Year	0.471	-0.323	-0.193	**20-Year**	0.263	0.360	0.176
4-Year	0.410	-0.034	-0.658	**25-Year**	0.257	0.369	0.267
10-Year	0.305	0.243	-0.271	**30-Year**	0.253	0.372	0.303

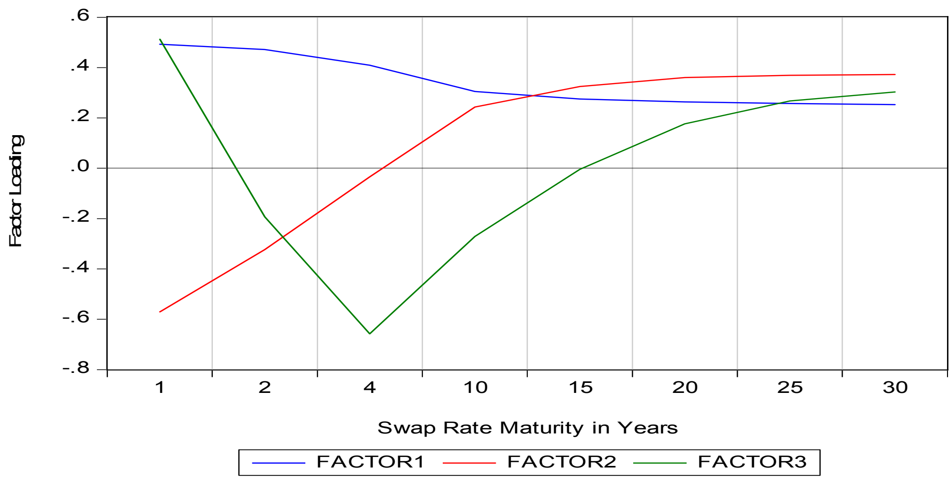

Figure 2: Factor Loadings

Based upon the patterns of the factor loadings for the first principal component, a shock to the first factor affects swap rates corresponding to each maturity in the same direction. A shock to the second factor affects swap wates corresponding to relatively shorter term maturities (i.e., 1 year Swap Rate, 2 year swap rate, and 4 year Swap Rate) in the opposite direction to returns corresponding to the longer term maturities (i.e., the 10 year swap rate out through the 30 year swap rate). Although it explains approximately 0.2% of the total variation in the group, we provide an interpretation for the third factor since it has a clear interpretation. Factor 3, presented in Figure 1, is a curvature factor; it shifts swap rates with relatively shorter term maturities and relatively longer-term maturities in the opposite direction (i.e., 1 year, 20 year, 25 year, and 30 year) from relatively middle-term maturities (i.e., 2 year, 4 year, 10 year, and 15 year).

With regards to our sample, the first two factors explain approximately 99.76% of the cumulative variation in the sample; with the first factor explaining approximately 93.16% of the variation in the sample and the second factor explaining about 6.60% of the variation in the swap rate sample. The remaining six factors would be regarded as noise. This highlights that PCA is a powerful tool that enables us to summarize the data with a smaller number of factors or variables.

4. Empirical Results

Estimates of the continuous time model are presented in Table 3. Turning to the one year rate the results imply a CKLS estimate of $\gamma = 0.2471$ indicating a low volatility-level effect for this rate which is significant. There is no evidence of mean reversion in the rate. For the two year rate the results imply a estimate of $\gamma = 0.1559$ which is significant and is no evidence of mean reversion in the rate. For the remaining rates the level-effects are of same magnitude as the two year rate.

Table 3: Gaussian Estimates of Continuous Time Swap Model

CKLS Model	α	β	σ^2	γ	Log-likelihood
1 Year					
	0.0000 (0.0000)	-0.0004 (0.0003)	0.0000 (0.0000)	0.2471 (0.0180)	25816.8
2 Year					
	0.0000 (0.0000)	-0.0005 (0.0004)	0.0000 (0.0000)	0.1559 (0.0032)	24791.5
4 Year					
	0.0000 (0.0000)	-0.0005 (0.0007)	0.0000 (0.0000)	0.1226 (0.0035)	24335.1
10 Year					
	0.0000 (0.0000)	-0.0009 (0.0009)	0.0000 (0.0000)	0.1264 (0.0038)	24278.9
15 Year					
	0.0000 (0.0000)	-0.0009 (0.0009)	0.0000 (0.0000)	0.1427 (0.0039)	24422.9
20 Year					
	0.0000 (0.0000)	-0.0009 (0.0009)	0.0000 (0.0000)	0.1522 (0.0039)	24518.7

Note: The parameter estimates with standard errors are presented for each model. 0.0000 denotes numbers less than 10^{-4}.

Having completed the analysis of the results from the continuous time model, we now examine those for the discrete time models. Beginning with the ARMA model, we find that, with the exception of the 1 year swap rates, where the best specification was an ARMA (1,1), the best specification for all other frequencies is an ARMA (1,0), as illustrated in Table 4.

Table 4: ARMA Model Results

	US1YS	US2YS	US4YS	US10YS	US15YS	US20YS	US25YS	US30YS
α	-2.254	-0.553	1.125	3.448	3.839	3.896	3.928	3.928
	(8.548)	(6.400)	(4.746)	(2.067)	(1.926)	(2.053)	(2.071)	(2.071)
$\rho(1)$	1.000	1.000	1.000	0.999	0.999	0.999	0.999	0.999
	(0.000)	(0.000)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
$\theta(1)$	0.082	-----	-----	-----	-----	-----	-----	-----
	(0.017)	(-----)	(-----)	(-----)	(-----)	(-----)	(-----)	(-----)
Log-Likelihood	6038.788	5084.298	4660.206	4613.227	4758.697	4862.897	4922.960	4922.960
AIC	-6.225	-5.690	-5.452	-5.425	-5.507	-5.565	-5.599	-5.599
SBIC	-6.218	-5.685	-5.447	-5.421	-5.502	-5.561	-5.595	-5.595

The result for the 1-year swap rates indicates that there is a significant moving average component in the prevailing swap rate today. One should further note that for all frequencies, there is a significant first-order autoregressive component in the determination of the current swap rate.

Table 5: ARIMA Model Results

	US1YS	US2YS	US4YS	US10YS	US15YS	US20YS	US25YS	US30YS
α	-0.001	-0.001	-0.001	-0.001	0.001	-0.001	-0.001	-0.001
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
$\rho(1)$	-----	-----	-----	0.750	-----	0.734	-----	0.738
	(-----)	(-----)	(-----)	(0.397)	(-----)	(0.275)	(-----)	(0.322)
$\theta(1)$	0.082	0.028	0.010	-0.761	-0.004	-0.752	-0.005	-0.753
	(0.017)	(0.017)	(0.017)	(0.389)	(0.017)	(0.267)	(0.017)	(0.314)
Log-Likelihood	6038.549	5085.447	4660.117	4611.578	4758.318	4861.995	4922.645	4964.769
AIC	-6.225	-5.690	-5.452	-5.425	-5.507	-5.565	-5.599	-5.623
SBIC	-6.220	-5.686	-5.447	-5.418	-5.502	-5.559	-5.594	-5.616

Given the fact that the unit root tests presented in Section 3 provided a strong indication that US swap rates were non-stationary, ARIMA models were estimated, where these results of the best models can be found in Table 5. The results here differ from those from the ARMA models in that for the 1 year, 2 year, 4 year, 15 year and 25 year swap rates, the best specification is found to be an ARIMA (0, 1, 1), with the best specification for all other frequencies being an ARIMA (1, 1, 1), although one should show some caution when interpreting the results for the 4 year, 15 year and 25 year swap rate results due to the lack of significance. This implies that for the 10 year, 20 year and 30 year swap rates there is a once again a significant first-order autoregressive component in the current swap rate determination, as opposed to the other frequencies. One should further note that there is a significant first-order moving average term for all data frequencies.

As stated previously, the underlying assumption of the ARMA and ARIMA models is that the underlying data series follows either a stationary or non-stationary process, respectively. An interesting approach would be to extend this by arguing the swap rates are fractionally integrated. In order to investigate this alternate hypothesis, ARFIMA models are estimated across all data series. The results from these models are presented in Table 6.

Table 6: ARFIMA Model Results

	US1YS	US2YS	US4YS	US10YS	US15YS	US20YS	US25YS	US30YS
α	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
$\rho(1)$	-----	-0.688	0.008	0.010	-0.949	0.014	0.012	0.013
	(-----)	(0.313)	(0.009)	(0.009)	(0.051)	(0.008)	(0.008)	(0.008)
$\rho(2)$	-----	-----	-----	-----	0.014	-----	-----	-----
	(-----)	(-----)	(-----)	(-----)	(0.003)	(-----)	(-----)	(-----)
$\theta(1)$	0.054	0.708	-----	-----	0.969	-----	-----	-----
	(0.017)	(0.313)	(-----)	(-----)	(0.048)	(-----)	(-----)	(-----)
$\theta(2)$	-----	0.034	-----	-----	-----	-----	-----	-----
	(-----)	(0.017)	(-----)	(-----)	(-----)	(-----)	(-----)	(-----)
Log-Likelihood	6042.867	5085.060	4660.191	4612.881	4757.230	4862.564	4922.576	4965.870
AIC	-6.227	-5.690	-5.452	-5.425	-5.506	-5.565	-5.599	-5.623
SBIC	-6.223	-5.681	-5.447	-5.421	-5.497	-5.561	-5.594	-5.619

One should again proceed with caution when interpreting the results for the 4 year, 10 year, 25 year and 30 year swap rates due to the lack of significance of the terms in the respective models. The results for the 4 year, 10 year, 20 year, 25 year and 30 year swap rates are identical to those from the ARMA models, with the results indicating that there is a significant first-order autocorrelation component in the prevailing swap rate, while the result for the 1 year swap rate is identical to the ARIMA model in exhibiting a significant first-order moving average component in the current swap rate. Interestingly, however, the results for the 2 year and 15 year swap rates indicate more persistence than exhibited by the other models, with the best specification for the 2 year swap rate being an ARFIMA (1, d, 2), which suggests that there is significant first-order autocorrelation component and a significant second-order moving average component in the prevailing swap rate. This implies that not only does the prevailing swap rate in the previous period play a significant role in determining the current swap rate, but any shocks to the swap rate over the preceding two periods are found to have a significant impact as well. There is similar persistence when examining the results for the 15 year swap rate, where the best model is an ARFIMA (2, d, 1), however, the persistence here is in the autocorrelation, as opposed to moving average, terms. This means that that both the previous two periods swap rates will have an impact on the current swap rate, while only shocks in the previous period will have any form of effect.

Forecast Results

Having estimated these models, ex-post dynamic forecasts were performed for each of the each of these models and the forecasts from all models were then compared using the Mean Absolute Percentage Error (MAPE) and Root Mean Squared Error (RMSE) forecast metrics, where these are calculated as follows:

$$MAPE = 100 \times \left\{ \frac{1}{M} \sum_{i=1}^{M} \left| \left(\frac{r_i^a - r_i^f}{r_i^a} \right) \right| \right\} \qquad RMSE = 100 \times \left\{ \frac{1}{M} \sum_{i=1}^{M} \left(r_i^a - r_i^f \right)^2 \right\}$$

where r_i^a denotes the actual observed value at time *i*, r_i^f denotes the forecasted value at time *i* and *M* denotes the forecast horizon.

Table 7: Forecast Metrics

Panel A - Forecasting Comparison Using the Mean Absolute Percentage Error

	US1YS	US2YS	US4YS	US10YS	US15YS	US20YS	US25YS	US30YS
ARMA	1.180%	0.991%	1.731%	7.193%	7.042%	6.178%	5.886%	5.428%
ARIMA	8.131%	8.198%	7.882%	12.928%	14.786%	15.730%	14.939%	16.051%
ARFIMA	8.002%	8.109%	7.880%	12.923%	14.781%	15.667%	14.940%	15.992%
CKLS	18.873%	21.027%	11.247%	3.099%	4.062%	4.665%	4.870%	5.211%

Panel B - Forecasting Comparison Using the Root Mean Squared Error

	US1YS	US2YS	US4YS	US10YS	US15YS	US20YS	US25YS	US30YS
ARMA	0.427	0.343	0.372	0.647	0.508	0.426	0.395	0.365
ARIMA	2.593	2.529	1.470	1.128	1.092	1.089	1.015	1.055
ARFIMA	2.546	2.477	1.470	1.128	1.090	1.087	1.015	1.052
CKLS	0.114	0.133	0.115	0.081	0.119	0.141	0.151	0.161

The results for these forecast metrics can be found in Table 7. Based on the RMSE the continuous time model have generally a better forecasting performance for the one, two, four and ten year rates compared to the discrete time models. At the longer end of the curve the CKLS model has a also smaller RMSE than the discrete time models. Based on the MAPE at the short end of the curve for the one, two and four year rates generally the discrete time models perform well. At the longer end of the curve the continuous time model has a satisfactory performance.

6. Conclusions

This paper has compared continuous and discrete time approaches to modelling and forecasting US swap rates for a range of maturities. Using daily data we compared the forecast performance of the continuous time CKLS with discrete time ARMA, ARIMA and ARFIMA models. We generally find that the continuous time model has a satisfactory performance across the curve. The PCA analysis indicates that the first two factors explain approximately 99.76% of the cumulative variation in the sample; with the first factor explaining approximately 93.16% of the variation in the sample and the second factor explaining about 6.60% of the variation in the swap rate sample.

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