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Skewness in energy returns: Estimation, testing and implications for tail risk

M. Angeles Carnero, Angel León, Trino-Manuel Ñíguez *

Dpto. Fundamentos del Análisis Económico (FAE), Universidad de Alicante, Spain, and Westminster Business School, University of Westminster, United Kingdom

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ABSTRACT

In this paper we estimate the skewness of the unconditional distribution of energy returns and test its statistical significance. We compare the performance of traditional and robust tests for skewness with those based on the implied unconditional skewness in a TGARCH model with Gram-Charlier (TGARCH-GC) innovations. We also analyze the implications of TGARCH-GC skewness for tail risk through evaluation of Value-at-Risk (VaR) and expected shortfall (ES) accuracy. Our results show that crude oil (Brent and WTI) and Gasoline returns are negatively skewed, while we do not find evidence of skewed distribution for other energy returns such as Heating oil, Kerosene and Natural gas. This indicates that the returns of the former are likely to encapsulate more largely the effect of negative shocks and so present higher tail risk than those of the latter. These results differ from traditional and robust tests for skewness providing important information on how to improve mean-variance risk management measures. Indeed, we find that the three-moment VaR and ES measures based on the third-order Cornish Fisher (CF3) expansion for the unconditional distribution of returns considerably improve their corresponding two-moment ones. We adopt CF3 to disentangle skewness effects from kurtosis in tail risk.

1. Introduction

Portfolio return skewness can be viewed as a source of tail risk and so, managing skewness risk becomes crucial since it measures the outcome resulting from bad news in financial markets that causes portfolio negative returns. The skewness importance in asset pricing was early recognized by Kraus and Litzenberger (1976), (1983) to model investor's behavior. Scott and Horvath (1980) show that a risk-averse investor with consistent moment preferences exhibits a positive preference for skewness.[1] In expected utility theory, preference for skewness captures the investor's gambling nature that is associated with prudence; see Kimball (1990) and Ebert and Wiesen (2011). Kumar (2009) shows evidence that people who find lotteries attractive are likely to invest much more in stocks with higher idiosyncratic volatility, higher skewness and lower prices, even if those stocks have lower expected returns. This empirical study does complement the skewness heterogeneity preference model by Mitton and Vorkink (2007) where investors with greater demand for skewness will, in equilibrium, hold less-diversified portfolios than those with less demand for skewness. Some recent works based on the role of skewness in portfolio choice are, for instance, Zakamouline and Koekebakker (2009),[2] Ghysels et al. (2016), De Roon and Karehnke (2017), Ñíguez et al. (2019) and Zhen and Chen (2022). For energy markets, in particular, Kuang (2021) evaluates clean energy portfolio risk through the four-moment modified Value-at-Risk (VaR) by Favre and Galeano (2002), which accounts for skewness and excess kurtosis, in order to build more diversified portfolios respecting a benchmark equity index portfolio.

* Corresponding author.
E-mail address: t.m.niguez@wmin.ac.uk (T.-M. Ñíguez).

[1] Consider an investor who cares about the payoff uncertainty of the asset return, r, via expected utility function, $E[U(\cdot)]$. Let μ, σ^2 and sk denote, respectively, the unconditional mean, variance and skewness of r. The investor's expectation of the third-order Taylor series expansion of $U(\cdot)$ around μ is given by $E[U(r)] \approx U(\mu) + \frac{1}{2}U^{(2)}(\mu)\sigma^2 + \frac{1}{6}U^{(3)}(\mu)(\sigma^2)^{3/2}sk$, where $U^{(i)}(\mu)$ is the derivative of order i of $U(\cdot)$ evaluated at point μ, and satisfying the following conditions: $U^{(2)}(\cdot) < 0$ (aversion to variance) and $U^{(3)}(\cdot) > 0$ (preference for positive skewness).

[2] Assume the investor has a wealth of W and invests a in the risky asset r and $W-a$ in the risk-free asset r_f. If we maximize expected utility in footnote 1 with respect to the investor's wealth after one period Δt, then the optimal amount invested in the risky asset a^* is given by $a^* \approx \frac{SR}{\lambda_2 \sigma\sqrt{\Delta t}}(1 + \lambda_3 \frac{sk}{2} SR)$ where $SR = \frac{(\mu - r_f)}{\sigma\sqrt{\Delta t}}$ denotes the Sharpe ratio, $\lambda_2 = -\frac{U^{(2)}(\mu)}{U^{(1)}(\mu)}$ is the traditional absolute risk aversion coefficient, and $\lambda_3 = -\frac{U^{(3)}(\mu)}{U^{(2)}(\mu)}$ is the coefficient of absolute prudence. See also Ñíguez et al. (2016).

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Testing for evidence of asymmetry in energy commodities, such as oil and oil-related products, is relevant since they tend to exhibit higher tail risk than stocks, see Aboura and Chevallier (2013) and Zhang et al. (2022). Price-formation fundamentals in energy markets are rather different than in the stock markets; see, e.g., Baur and Dimpfl (2018). In fact, commodity prices are highly sensitive to global economic issues, so they present more accentuated sharp falls, which might make energy returns skewness prevalent in tail risk. This discussion motivates us to focus on the skewness in energy markets which has been less studied than that of the stock market.

The Brent, WTI and Gasoline spot returns series, that we study here among others, may be skewed due to several factors: (i) Sudden shifts in either supply or demand can result in significant uneven price changes because of, on the one hand, geopolitical tensions, natural disasters, and production policies affecting the supply side, on the other hand, changes in consumption patterns or economic growth have a more prevalent effect on the demand side; (ii) limited storage capacity can give rise to notable fluctuations in prices. A sudden increase in supply without sufficient storage capacity can lead to lower prices, while higher prices can arise from a sudden drop in supply; (iii) seasonality does also affect the prices of these commodities. For instance, the demand for crude oil and Gasoline increases in summer, when people drive more, and decreases in winter when they travel less; (iv) disruptions in refining operations can cause the price of Gasoline to fluctuate. For example, hurricanes and other natural disasters can disrupt refineries, reducing Gasoline production and increasing prices; and (v) fluctuations in exchange rates. Since these commodities are priced in US dollars, any changes in exchange rates can affect their prices. A strong (weak) dollar can cause prices to fall (rise).

Skewness is typically estimated by the sample skewness which has been shown to be highly unreliable and biased; see, e.g., Kim and White (2004), Bai and Ng (2005), Ghysels et al. (2016) and Li (2020). To address this issue, robust estimators of the skewness in Bowley (1901), Yule (1911) and Hinkley (1975) have been applied in previous studies. Despite that, many papers still employ the sample estimator of Pearson's moment of skewness. See, for instance, Fernandez-Perez et al. (2018), Jondeau et al. (2019), Mo et al. (2019), Dai et al. (2021) and Liu et al. (2021).

In regard to testing for skewed financial returns, several methods have been proposed in the literature delivering mixed results; see, for example, Peiró (1999), Kim and White (2004), Premaratne and Bera (2005) and Bai and Ng (2005). In spite of that, many of the existing conditional heteroskedastic models assume a symmetric distribution for the innovations, which implies, for most of these models, that returns are symmetric.[3] During the last years, models assuming asymmetric distributions for the innovations have become more popular since they allow to test for the significance of the skewness parameter.[4] Regarding energy markets, Aloui and Mabrouk (2010), Cheng and Hung (2011), Lyu et al. (2017), Laporta et al. (2018) and Kuang (2022), among others, fit GARCH-type models to energy return series and compare their performance for measuring VaR under alternative distributions. Their results show the importance of accounting for skewness and kurtosis in energy commodity returns since models with skewed and fat-tailed distributed innovations outperform those under symmetric returns distributions. Nevertheless, in such studies where GARCH-type models are used together with non-normal distributed innovations, the unconditional skewness cannot be estimated or tested, remaining sample skewness measures as the typical way used to estimate and test the returns unconditional distribution's skewness. To the best of our knowledge, skewness in those models is not known and hence, it has not been used for asset pricing and risk management. The complication stems from that the GARCH-type structure interacts with the unconditional skewness of the distribution, so the derivation of the model implied skewness becomes cumbersome. This problem has been recently studied in Carnero et al. (2022) for the Threshold GARCH (TGARCH) model of Zakoian (1994) assuming a Gram-Charlier distribution, see Jondeau and Rockinger (2001), for the innovations (henceforth, TGARCH-GC).[5]

In this paper, we investigate the appropriateness of alternative measures of skewness for energy return series in relation to the skewness implied in a GARCH-type model with non-normal errors. The main contributions are threefold. First, we apply the TGARCH-GC model to estimate the returns' implied (unconditional) skewness and compare it with sample and robust measures of skewness.[6] Second, we show that testing for skewed returns through the statistical significance of the implied skewness of a TGARCH-GC model differs from traditional skewness tests based on sample skewness or quantiles; see, for example, Cabilio and Masaro (1996), Bai and Ng (2005), Ngatchou-Wandji (2006) and Ekström and Jammalamadaka (2012). We do this through the methodology recently proposed in Carnero et al. (2022). This approach allows to estimate and test for the significance of energy returns unconditional skewness in a feasible manner. Thus, our paper contributes to the discussion on the inference that can be made using different estimators of the skewness in energy markets. An analysis of the power functions performance for the previous skewness tests (both asymptotic and bootstrap ones) is also provided. In summary, the paper shows how to improve accuracy in measuring and testing for skewness in energy returns using techniques recently proposed in the literature.

Last but not least, we also analyse the implications of employing different measures of skewness for tail risk through the evaluation of VaR and expected shortfall (ES) accuracy. We estimate sample and implied VaR and ES measures by assuming the third-order Cornish-Fisher (CF3) expansion, see Cornish and Fisher (1938), for the unconditional distribution of returns. We study in some detail CF3, which has hardly been studied in the literature. The CF3 expansion depends on one parameter, which is closely related to the unconditional skewness for a given range of the parameter values. This parameter is replaced by the closed-form expression for the skewness implied in the TGARCH-GC model. We adopt CF3 instead of the popular CF4 (fourth order Cornish-Fisher) expansion so as to disentangle the unconditional effect of skewness in tail risk from the kurtosis performance under CF4. Definitively, we are only interested in the skewness marginal effects on the tail risk contribution with respect to the one based on the first two moments; that is why testing skewness in a previous stage is important.

[3] If the conditional mean is not constant, returns can be asymmetric although innovations are symmetric; see He et al. (2008).

[4] See, for example, Harvey and Siddique (1999), León et al. (2005), Ghysels et al. (2016), Carnero et al. (2022) and Serna (2022).

[5] More precisely, the TGARCH model has the advantage that its moments (both odd and even) can be computed and therefore, it is possible to derive their closed-form expressions. This is because the model is specified for the conditional volatility σ_t, and not for σ_t^2 as in other GARCH-type models. Hence, we can only obtain second-order Taylor approximations for the unconditional skewness measure when modeling directly σ_t^2. See, for instance, Alexander et al. (2021) for the case of the GJR-GARCH model of Glosten et al. (1993).

[6] When simulating 1000 series of daily returns from a TGARCH-GC model of sample size $T = 10000$, the sample estimates of the variance are close to the true variance, while the sample estimates of the skewness are very bad, even when the sample size is very large ($T = 100000$). As expected, the estimates of the implied skewness from the model are much better. Obviously, in this case, the model we are assuming for the data generating process (DGP) is the true one, and we are aware that in practice we do not know the true DGP. However, estimating the skewness assuming the above model, which captures many stylized facts of financial returns, seems to be a good alternative to standard methods in this case. Since daily financial returns are leptokurtic and not independent, we know that standard methods to estimate skewness are not going to perform well. More details are available upon request.

Finally, our results can be summarized as follows. First, we find evidence of negative skewness for the series of crude oil (WTI and Brent) and Gasoline returns, while the unconditional distribution of other energy returns such as Heating oil, Kerosene and Natural gas is found to have zero skewness.[7] These series with significant skewed distributions may confirm price fluctuations derived from some of the aforementioned economic factors, nevertheless the question of which of those factors can really cause skewed distributions deserves a deeper analysis that is beyond the scope of this work although welcoming for future research. Regarding the less skewed returns evidence found in Kerosene, Heating oil, and Natural gas, it may be attributed to several reasons: (i) relative homogeneity, their similar properties and uses lead to less variation in prices and returns as compared to commodities that have more diverse uses; (ii) there is some seasonality in the demand for Heating oil and Natural gas, although it is less pronounced compared to crude oil and Gasoline. Moreover, the demand for Kerosene is relatively stable throughout the year due to its use in aviation and other industrial applications; (iii) unlike crude oil and Gasoline, Natural gas and Heating oil have significant storage capacity, which helps to smooth out short-term fluctuations in supply and demand; (iv) while crude oil and Gasoline are often affected by geopolitical tensions and conflicts, the production and transportation of Kerosene, Heating oil, and Natural gas are often less affected. This stability in supply helps to stabilize prices and returns; (v) Natural gas and Heating oil are often sold under long-term contracts that can span several years, ensuring more stable prices and returns.

Second, the three-moment VaR and ES measures perform significantly better than their corresponding two-moment counterparts. These findings highlight that energy market asset pricing and risk management analyses that use sample skewness may deliver misleading conclusions regarding portfolio performance. Certainly, improved skewness measures, based on the TGARCH-GC implied unconditional skewness, does enhance the performance of skewness-based portfolios in terms of Sharpe ratio, see Zakamouline and Koekebakker (2009) and Li (2020) for further evidence on this issue.

The remainder of the paper is divided into four sections. Section 2 reviews the alternative methods for testing the unconditional skewness. In particular, we describe traditional tests for skewness as well as the TGARCH-GC model that will be used to estimate the unconditional skewness of returns, and also to test its statistical significance. Section 3 is about modeling the tail risk based on VaR and ES according to the CF3 expansion for the unconditional distribution of returns. In Section 4, we describe the data and apply the different methods to test whether energy returns are symmetric or skewed, and also the effects of skewness in tail risk through estimations of both VaR and ES measures. Section 5 concludes the paper.

2. Testing for skewness

We start by considering three classical statistics to test for skewness in energy returns. Specifically, the robust skewness statistics of Bowley (1901) and Yule (1911) as well as the sample skewness statistic. Next, we employ the TGARCH-GC model, for which we know the analytical expression of skewness, to estimate and test whether the unconditional skewness of energy returns is equal to zero.

2.1. Traditional tests for skewness

Consider a stationary time series, y_t, with mean μ and variance σ^2. It is well known (see, for example, Kendall & Stuart, 1969 that, if y_t with $t = 1, 2, \ldots, T$, is independent and identically distributed (*iid*) as a normal distribution, $\widehat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t$ is the sample mean and $\widehat{sk}$ is the sample skewness given by

$$\widehat{sk} = \frac{\frac{1}{T}\sum_{t=1}^{T}(y_t - \widehat{\mu})^3}{\left(\sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(y_t - \widehat{\mu})^2}\right)^3}, \tag{1}$$

then

$$\sqrt{T}\,\widehat{sk} \longrightarrow_d N(0, 6). \tag{2}$$

This skewness test is widely used by practitioners. However, as discussed by Bai and Ng (2005) and Holgersson (2007), among others, it has serious drawbacks. For example, it does not discriminate between skewed and non normal symmetric distributions. Several alternatives have been proposed and analyzed in the literature, for example, the robust skewness estimators of Bowley (1901) and Yule (1911) given, respectively, by

$$\widehat{sk}_B = \frac{\widehat{Q}_3 - 2\widehat{M} + \widehat{Q}_1}{\widehat{Q}_3 - \widehat{Q}_1} \tag{3}$$

and

$$\widehat{sk}_Y = \frac{\widehat{\mu} - \widehat{M}}{\widehat{\sigma}}, \tag{4}$$

where M is the median of y_t, σ is the standard deviation, Q_1 and Q_3 are the first and third quartiles respectively, and $\widehat{M}$, $\widehat{\sigma}$, $\widehat{Q}_1$ and $\widehat{Q}_3$ are their sample estimators. According to Ngatchou-Wandji (2006) and Ekström and Jammalamadaka (2012), it can be seen that

$$\sqrt{T}(\widehat{Q}_3 - 2\widehat{M} + \widehat{Q}_1) \longrightarrow_d N(0, \Theta^2), \tag{5}$$

where

$$\Theta^2 = \frac{0.25 \times 0.75}{f^2(Q_3)} + \frac{0.25 \times 0.75}{f^2(Q_1)} + \frac{1}{f^2(M)} + \frac{2 \times 0.25^2}{f(Q_3) \times f(Q_1)} - \frac{2 \times 0.25}{f(Q_3) \times f(M)} - \frac{2 \times 0.25}{f(Q_1) \times f(M)} \tag{6}$$

and f is the density function of y_t. Moreover, Cabilio and Masaro (1996) showed that

$$\sqrt{T}\,\widehat{sk}_Y \longrightarrow_d N(0, \sigma^{-2}\nu^2), \tag{7}$$

where

$$\nu^2 = \sigma^2 + \frac{1}{4f^2(M)} - \frac{E|y_t - M|}{f(M)}. \tag{8}$$

Using the previous asymptotic distributions, we could test whether or not the unconditional skewness of energy returns is equal to zero by means of three alternative tests statistics, which are all approximately distributed as a standard normal. More precisely, the three test statistics are given by

$$z_s = \frac{\sqrt{T} \times \widehat{sk}}{\sqrt{6}}, \quad z_B = \frac{\sqrt{T} \times (\widehat{Q}_3 - \widehat{Q}_1) \times \widehat{sk}_B}{\widehat{\Theta}} \quad \text{and} \quad z_Y = \frac{\sqrt{T} \times \widehat{sk}_Y}{\widehat{\sigma}^{-1}\,\widehat{\nu}}, \tag{9}$$

where Θ and ν are estimated by considering the sample estimators of M, σ, Q_1 and Q_3, as well as the sample counterpart of $E|y_t - M|$. Finally, the density function of y_t, denoted as f, can be estimated nonparametrically by using the Epanechnikov kernel function.

2.2. Tests for the TGARCH-GC skewness

First, we introduce the model for returns and the unconditional moments given in Carnero et al. (2022). Second, we consider several tests for the implied unconditional skewness under the TGARCH-GC model.

[7] In contrast to what one would conclude using sample skewness measures.

2.2.1. Model for returns

Consider a conditionally heteroskedastic process r_t given by

$$r_t = \mu + \varepsilon_t, \qquad \varepsilon_t = \sigma_t z_t, \tag{10}$$

where μ and σ_t^2 denote, respectively, the conditional mean and variance of r_t given the information set $\mathscr{F}_{t-1}$ and $\{z_t\}$ is a sequence of *iid* random variables distributed as a Gram-Charlier (GC) with probability density function (pdf) given by

$$h(x, \vartheta) = \phi(x)\varphi(x, \vartheta), \tag{11}$$

where $x \in \mathbb{R}$, $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{R}^2$ is the parameter vector, $\phi(\cdot)$ is the pdf of the standard normal distribution and $\varphi(\cdot)$ is defined as

$$\varphi(x, \vartheta) = 1 + \frac{\vartheta_1}{\sqrt{3!}}H_3(x) + \frac{\vartheta_2}{\sqrt{4!}}H_4(x), \tag{12}$$

such that $H_k(\cdot)$ are the k-th orthonormal Hermite polynomials. Specifically,

$$H_3(x) = (x^3 - 3x)/\sqrt{3!}, \quad H_4(x) = (x^4 - 6x^2 + 3)/\sqrt{4!}. \tag{13}$$

In this case, the third and fourth order moments of z_t are its skewness and kurtosis and will be denoted, respectively, by $sk_z = \vartheta_1$ and $k_z = 3 + ek_z$, being $ek_z = \vartheta_2$ the excess kurtosis, see Jondeau and Rockinger (2001). Consider $\psi_k = E[z_t^k] = \int_{-\infty}^{+\infty} x^k h(x, \vartheta)dx$ where $h(\cdot)$ is given in (11), then $\psi_1 = 0$, $\psi_2 = 1$, $\psi_3 = sk_z$ and $\psi_4 = 3 + ek_z$.

We assume that the error process $\{\varepsilon_t\}$ in (10) follows the TGARCH (1,1) model proposed by Zakoian (1994) which specifies directly the volatility σ_t and, as discussed by Rodríguez and Ruiz (2012), it is an appropriate and flexible GARCH-type model to represent the dynamic properties of financial returns, namely, excess kurtosis, conditional heteroskedasticity and leverage effect. In this model, σ_t is given by

$$\begin{aligned}\sigma_t &= \omega + \beta\sigma_{t-1} + \alpha^+\varepsilon_{t-1}^+ - \alpha^-\varepsilon_{t-1}^- \\ &= \omega + \sigma_{t-1}c_{t-1},\end{aligned} \tag{14}$$

such that $c_t = \beta + \alpha^+ z_t^+ - \alpha^- z_t^-$, $\omega > 0$, $\beta \geq 0$, $\alpha^+ \geq 0$ and $\alpha^- \geq 0$. We use the notation $x_t^+ = \max(x_t, 0)$ and $x_t^- = \min(x_t, 0)$ where x_t can be either ε_t or z_t.

This model allows for an asymmetric response of volatility to positive and negative past returns. In particular, the volatility tends to be higher following negative return shocks than following positive ones of the same magnitude. This leads generally to negative cross-correlations between lagged returns and volatility. As we can see in (14), when ε_{t-1} is positive, the volatility response is linear in ε_{t-1} with slope α^+, but if ε_{t-1} is negative, the slope of the response is α^- and it is expected that $\alpha^+ < \alpha^-$. Notice that when $\alpha^+ = \alpha^-$, the volatility responds symmetrically to positive and negative past returns and the model collapses to the Absolute Value GARCH (AVGARCH) model of Taylor (1986) and Schwert (1989).

Francq and Zakoian (2010) show that (14) is strictly stationary if $E(\ln(c_t)) < 0$, and it is second-order stationary if $E(c_t^2) < 1$. Carnero et al. (2022) show that the skewness of r_t in (10) is

$$sk_r = sk_z \frac{E(\sigma_t^3)}{E(\sigma_t^2)^{3/2}}, \tag{15}$$

where

$$E(\sigma_t^k) = \frac{\omega^k f_k}{\prod_{j=1}^k (1 - a_j)}, \qquad k = 1, 2, 3 \tag{16}$$

with $f_1 = 1$, $f_2 = 1 + a_1$, $f_3 = 1 + 2a_1 + 2a_2 + a_1 a_2$, and $a_k = E(c_t^k)$ is given by

$$\begin{aligned} a_1 &= \beta + g_1\phi_0(1 - ek_z/24), \\ a_2 &= \beta^2 + (\alpha^+)^2 - 2\beta g_1\psi_1^- + t_2\psi_2^-, \\ a_3 &= \beta^3 + 3\beta(\alpha^+)^2 + (\alpha^+)^3 sk_z - 3\beta^2 g_1\psi_1^- + 3\beta t_2\psi_2^- - g_3\psi_3^-, \end{aligned} \tag{17}$$

where $\psi_1^- = (ek_z\phi_0 - 24\phi_0)/24$, $\psi_2^- = (3 - 2sk_z\phi_0)/6$, $\psi_3^- = (2sk_z - ek_z\phi_0 - 8\phi_0)/4$ and $\phi_0 = 1/\sqrt{2\pi}$. Note that $\psi_k^- = E[(z_t^-)^k] = \int_{-\infty}^0 x^k h(x, \vartheta)dx$.

Finally, other GARCH-type models are based on modeling directly the conditional variance σ_t^2. For instance, if we consider the GJR-GARCH of Glosten et al. (1993), there is no closed-form expression for $E(\sigma_t^3)$ in (15) and so, $E(\sigma_t^3)$ is obtained according to the following second-order Taylor approximation: $E(\sigma_t^3) \approx \frac{5}{8}[E(\sigma_t^2)]^{3/2} + \frac{3}{8}E(\sigma_t^4)[E(\sigma_t^2)]^{-1/2}$, where $\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha_1^+(\varepsilon_{t-1}^+)^2 + \alpha_1^-(\varepsilon_{t-1}^-)^2$, $\varepsilon_t = \sigma_t z_t$ and z_t is GC-distributed (GJR-GC model). This is the main reason why we have not used alternative GARCH models.[8]

2.2.2. Testing the TGARCH-GC implied unconditional skewness

A test for skewness of the unconditional distribution of returns can be obtained through the asymptotic distribution of the maximum likelihood (ML) estimator $\widehat{sk}_r$ of the unknown sk_r in (15) by using the Delta method. Nevertheless, this estimator leads to a very slow convergence rate to its asymptotic distribution. This evidence can be found, for instance, in Francq and Zakoian (2022) when testing the existence of the unconditional GARCH moments. They tackle their slow convergence rate problem by implementing bootstrap-based tests. In our case, this latter method has the drawback of being very high time-consuming due to the large number of TGARCH-GC estimations that it requires.

Another alternative testing method is driven by Anatolyev (2019), who implements the IM test of Ibragimov and Müller (2010), to build alternative confidence intervals (CIs) from those based on the large sample theory for the unconditional GARCH kurtosis. The IM method allows to construct CIs for sk_r with a small number of repeated TGARCH-GC estimations. To compute the IM test statistic, the original sample is divided into $q \geq 2$ non-overlapping groups, with n_j observations in each group j and $T = \sum_{j=1}^q n_j$. Let us denote by $\widehat{sk}_{r,j}$ the implied skewness estimate of sk_r obtained with only the observations in group j. The robust test for the null hypothesis H_0: $sk_r = 0$ against the two-sided alternative H_1: $sk_r \neq 0$ is based on the usual t-statistic:

$$t_{sk_r} = \sqrt{q}\,\frac{\overline{\widehat{sk}}_r}{S}, \tag{18}$$

where $\overline{\widehat{sk}}_r = q^{-1}\sum_{j=1}^q \widehat{sk}_{r,j}$ and $S^2 = (q-1)^{-1}\sum_{j=1}^q(\widehat{sk}_{r,j} - \overline{\widehat{sk}}_r)^2$. The IM test statistic is approximately distributed as a Student-t with $q - 1$ degrees of freedom. The IM confidence interval for sk_r with $1 - \alpha$ approximate coverage can be constructed as $\overline{\widehat{sk}}_r \pm cv_\alpha \frac{S}{\sqrt{q}}$, where cv_α denotes the $(1 - \alpha/2)$ quantile of the Student-t distribution with $q - 1$ degrees of freedom. The null hypothesis, H_0: $sk_r = 0$, will be rejected when the zero is not contained in the IM confidence interval for sk_r.

3. Skewness implications for tail risk

We consider the CF3 expansion for the unknown returns unconditional distribution in (10) to compute quantiles for VaR and ES calculation and evaluate the implications of skewness in returns' tail risk.

[8] Indeed, we tried to compute estimations of GJR-GC approximated skewness (also considering other distributions) for both daily and weekly series, and in most cases, the condition for the existence of the third order moment did not hold leading to infinite skewness. For those cases where it was possible to compute the approximate skewness, we found large differences with respect to the TGARCH-GC skewness. In short, the exact skewness under the TGARCH model seems to behave better than the approximated one under the GJR for the parameter values obtained with the energy return series of this paper.

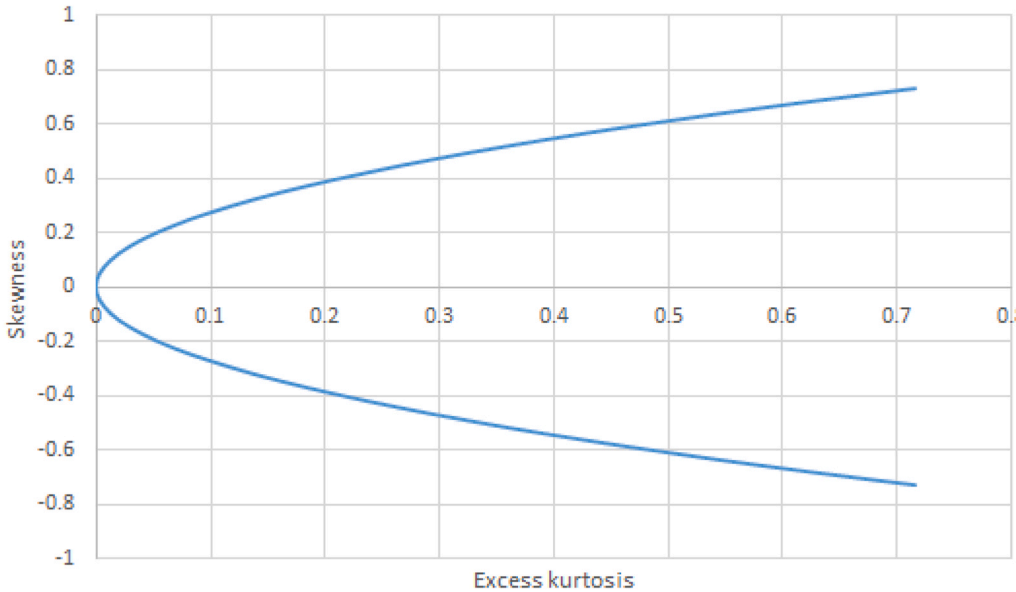

Fig. 1. Skewness-Excess kurtosis frontier for CF3 distribution.

3.1. Third-order Cornish-Fisher expansion

Let y denote the CF3 random variable, which is a second-order polynomial expansion of the standard normal w given by

$$y = \varphi(w) = w + \frac{\xi}{6}(w^2 - 1), \tag{19}$$

with mean and variance: $\mu_y = 0$ and $\sigma_y^2 = 1 + \xi^2/18$. The skewness is $sk_y = E(y^3)/(\sigma_y^2)^{3/2}$ where $E(y^3) = (\xi/3)^3 + \xi$. To compute the quantiles of y, we must guarantee the monotonicity condition, which means that the transformation from w to y is one-to-one. This implies that the derivative of y relative to w is positive, $\varphi'(w) > 0$. The quantile of y in (19) at a given probability level α is:

$$y_\alpha = \varphi(w_\alpha) = w_\alpha + \frac{\xi}{6}(w_\alpha^2 - 1), \tag{20}$$

with $w_\alpha = \Phi^{-1}(\alpha)$ such that $\Phi(\cdot)$ denotes the cumulative distribution function (cdf) of the standard normal distribution. It is verified that $sk_y \approx \xi$ when $\xi \in (-3/4, 3/4)$.[9] This result suggests that the sample skewness of returns series (with constant mean) can be a good candidate estimate for the parameter ξ. The fourth-order moment of y can be easily obtained as $E(y^4) = (1 + c_2\xi^2 + c_4\xi^4)$ with $c_2 = 5/6$ and $c_4 = 5/324$. The kurtosis k_y, defined as $k_y = E(y^4)/(\sigma_y^2)^2$, is an even function with respect to ξ such that $3 \le k_y < 3.72$. Fig. 1 exhibits the skewness-excess kurtosis frontier of CF3, i.e. the points $(ek_y(\xi), sk_y(\xi))$ where $ek_y = k_y - 3$ for the previous range of ξ. The plot shows that the CF3 allows ranges of skewness and excess kurtosis that include those of the return series considered in our empirical analysis.

Consider the affine transformation $x = a + by$ with y as the random variable in (19), and both a and b as the location and scale parameters, respectively. The cdf of x is given by $F_x(u; \xi) = \Phi(\gamma(u, \xi))$ with $u \in \mathbb{R}$ and $\gamma(u, \xi) = \varphi^{-1}(\frac{u-a}{b})$; while the pdf of x is obtained as $f_x(u; \xi) = \frac{\phi(\gamma)}{b\varphi'(\gamma)}$. Note that $v = \gamma$ is one of the roots of the equation $\varphi(v) = \frac{u-a}{b}$ verifying that $\varphi'(\gamma) > 0$. The previous equation can be rewritten as $h(v) = 0$ such that $h(v) = (\xi/6)v^2 + v - (\bar{u} + \xi/6)$ where $\bar{u} = (u - a)/b$. Then, $\gamma(u, \xi) = -3\xi^{-1}(1 - \sqrt{\Delta})$ with $\Delta = (\xi/3)^2 + 2\bar{u}(\xi/3) + 1$.

[9] This result is very easy to check. It is also available upon request.

3.2. VaR under CF3

Definitively, the unconditional VaR at probability level α of r_t in (10) can be obtained through y_α in (20) such that ξ is replaced with sk_r in (15), as follows

$$VaR3(\alpha) = \mu + \frac{\sigma_r}{\sigma_y}\left(w_\alpha + \frac{1}{6}(w_\alpha^2 - 1)sk_r\right) \tag{21}$$

where, according to (16), $\sigma_r^2 = \omega^2(1 + a_1)(1 - a_1)^{-1}(1 - a_2)^{-1}$ with a_k in (17). Henceforth, we denote (21) as the implied three-moment VaR (VaR3). We also compare VaR3 with the two-moment VaR (VaR2) based on the standard normal distribution. Hence, the implied VaR2 is easily obtained from (21) when $sk_r = 0$ (which implies $\sigma_y = 1$), i.e.

$$VaR2(\alpha) = \mu + \sigma_r w_\alpha. \tag{22}$$

3.3. ES under CF3

Following Maillard (2018), we can easily obtain a closed-form expression for the conditional VaR, or ES, under the CF3 expansion in (19) as

$$\begin{aligned} E[y \mid y \le y_\alpha] &= E[\varphi(w) \mid w \le w_\alpha] \\ &= -\frac{\phi(w_\alpha)}{\alpha}\left(1 + \frac{\xi}{6}w_\alpha\right). \end{aligned} \tag{23}$$

Fig. 2 exhibits, on the left panel, the ES in (23) and the corresponding VaR, or quantile y_α, in (20) with $w_\alpha = -2.326$ for $\alpha = 1\%$ as a function of $\xi \in (-3/4, 3/4)$. The right panel displays the cdf of y evaluated at each ES value[10] represented in the left plot according to ξ.

The implied unconditional ES of r_t in (10) under CF3 is the conditional expectation of returns smaller than VaR in (21) at probability level α,

$$ES3(\alpha) = \mu + \frac{\sigma_r}{\sigma_y}E[y \mid y \le y_\alpha], \tag{24}$$

[10] The cdf at point y^*_α, which denotes the ES in (23), is $F_y(y^*_\alpha; \xi) = \Phi(-3\xi^{-1}(1 - \sqrt{\Delta}))$ with $\Delta = (\xi/3)^2 + 2y^*_\alpha(\xi/3) + 1$. For more details, see Section 3.1.

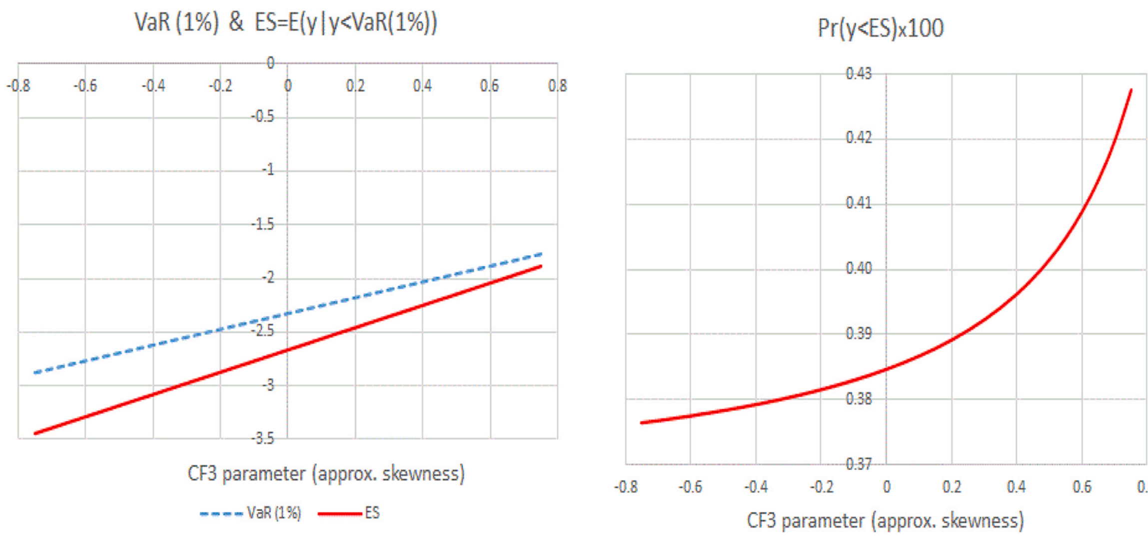

Fig. 2. The left panel provides plots of one percent VaR in (20) and corresponding ES in (23). The right panel exhibits the CF3 cdf evaluated at the ES values (see footnote 4) displayed in the left plot. The CF3 parameter in the x-axis is ξ in (19).

Table 1
Summary statistics for log returns.

	WTI	Brent oil	Heating oil	Gasoline	Kerosene	Natural gas
Start date	01/02/1986	02/06/1986	02/06/1986	02/06/1986	01/01/1990	07/01/1997
End date	05/04/2021					
Panel 1: Daily frequency						
T_d (obs.)	8883	8600	8758	8760	7793	6097
Mean	0.01	0.01	0.02	0.02	0.02	− 0.01
Std. dev.	2.78	2.55	2.51	2.69	2.60	5.12
Min	− 42.36	− 64.37	− 47.01	− 30.06	− 37.76	− 102.51
Max	42.58	41.20	22.95	23.51	32.64	74.56
Skewness	− 0.52	− 1.83	− 1.38	− 0.47	− 0.58	− 0.19
Kurtosis	34.95	72.51	35.75	14.62	19.72	59.39
Panel 2: Weekly frequency						
T_w (obs.)	1839	1768	1817	1817	1617	1263
Mean	0.05	0.07	0.08	0.08	0.07	− 0.03
Std. dev.	7.21	4.60	4.44	4.95	4.57	9.23
Min	− 180.17	− 40.78	− 29.14	− 44.80	− 30.51	− 144.90
Max	155.43	32.38	36.17	37.36	22.13	96.72
Skewness	− 3.08	− 0.65	0.04	− 0.51	− 0.36	− 1.92
Kurtosis	331.14	12.32	9.72	11.39	8.11	65.28

where ξ in (23) is replaced with sk_r in (15). The implied ES with only the first two moments is obtained when $sk_r = 0$ (and so, $\sigma_y = 1$), i.e.

$$ES2(\alpha) = \mu - \sigma_r \frac{\phi(w_\alpha)}{\alpha}. \qquad (25)$$

4. Empirical application

In this section we analyze empirically whether or not the unconditional distribution of energy returns is skewed. First, we consider different series of daily and weekly returns and test for skewness using the different methods described in Section 2. Second, we show practical applications of TGARCH-GC implied skewness for measuring tail risk.

4.1. Data and summary statistics

The series analyzed are the log returns computed as $r_t = 100\ln(P_t/P_{t-1})$ from samples of closing prices observed at daily frequency, $\{P_t\}_{t=1}^{T_d}$ with T_d as sample size, and weekly frequency, $\{P_t\}_{t=1}^{T_w}$ with T_w as sample size, of six common energy commodities traded in the market such as, West Texas Intermediate (WTI) crude oil; Brent crude oil; Heating oil No. 2; New York Harbour Conventional Gasoline Regular; Kerosene-Type jet fuel and Henry Hub Natural gas. Table 1 provides the returns summary statistics. The observation period for all series covers until April 5, 2021, and the start date for each series depends on data availability and is provided in Table 1. The data is publicly available from the Federal Reserve Economic Database (FRED) https://fred.stlouisfed.org/. All series present a negative skewness coefficient, except the weekly Heating oil returns, ranging from − 1.83 (Brent oil) to − 0.19 (Natural gas) for daily frequency, and from − 3.08 (WTI) to 0.04 (Heating oil) for weekly returns. The kurtosis is high for all series ranging from 14.62 (Gasoline) to 72.51 (Brent oil) for the daily series, and from 8.11 (Kerosene) to 331.14 (WTI) for weekly returns. The non-reported Jarque-Bera test rejects normality for all series.

4.2. Tests for skewness

We test for skewed distributions using, first, the classical tests described in Section 2.1 and later, we use the TGARCH-GC model described in Section 2.2.

Table 2
This table reports asymptotic and bootstrap p-values for the z-statistics in (26) from standard, Bowley and Yule tests for skewness, denoted as $p\text{-}v_a$ and $p\text{-}v_b$, respectively.

		WTI	Brent oil	Heating oil	Gasoline	Kerosene	Natural gas
Panel 1: Daily frequency							
standard	$p\text{-}v_a$	0.00***	0.00***	0.00***	0.00***	0.00***	0.00***
	$p\text{-}v_b$	0.21	0.05*	0.05*	0.13	0.12	0.79
Bowley	$p\text{-}v_a$	0.26	0.25	0.05	0.10	0.07*	0.02***
	$p\text{-}v_b$	0.32	0.33	0.13	0.31	0.17	0.10
Yule	$p\text{-}v_a$	0.02**	0.56	0.44	0.46	0.54	0.89
	$p\text{-}v_b$	0.01**	0.59	0.50	0.62	0.62	0.88
Panel 2: Weekly frequency							
standard	$p\text{-}v_a$	0.00***	0.00***	0.49	0.00***	0.00***	0.00***
	$p\text{-}v_b$	0.33	0.15	0.89	0.36	0.27	0.21
Bowley	$p\text{-}v_a$	0.02**	0.10	0.24	0.38	0.02**	0.02**
	$p\text{-}v_b$	0.03**	0.16	0.25	0.50	0.03**	0.16
Yule	$p\text{-}v_a$	0.12	0.01***	0.36	0.08*	0.03**	0.89
	$p\text{-}v_b$	0.08*	0.02**	0.36	0.15	0.06*	0.84

Finally, (***) indicates rejection of the null hypothesis of symmetry at 1% level; (**) at 5% level and (*) at 10% level.

4.2.1. Traditional tests

We have computed standard, Bowley and Yule tests statistics given by (9).[11] Under the standard normal distribution, we have calculated both asymptotic and bootstrap p-values of the three tests. For the bootstrap approach, see Jiang et al. (2020), we have obtained 500 block-bootstrap time series, $r^*{}_t$, from the original series, r_t. For each of the bootstrap series $\{r^*{}_{t,i}\}$ where $i = 1, \ldots, 500$, we have found the three estimators of the skewness, $\widehat{sk^*}_i$, $\widehat{sk^*}_{B,i}$, and $\widehat{sk^*}_{Y,i}$ in (1), (3) and (4) respectively. Then, we calculated the sample variance of each bootstrap estimator with the 500 values previously obtained. For instance, $\widehat{Var}(\widehat{sk^*}) = \frac{1}{499}\sum_{i=1}^{500}(\widehat{sk^*}_i - \overline{\widehat{sk^*}})^2$ where $\overline{\widehat{sk^*}} = \frac{1}{500}\sum_{i=1}^{500}\widehat{sk^*}_i$. Finally, the bootstrap test statistics are given by

$$z^*{}_S = \frac{\widehat{sk}}{\sqrt{\widehat{Var}(\widehat{sk^*})}}, \quad z^*{}_B = \frac{\widehat{sk}_B}{\sqrt{\widehat{Var}(\widehat{sk^*}_B)}} \quad \text{and} \quad z^*{}_Y = \frac{\widehat{sk}_Y}{\sqrt{\widehat{Var}(\widehat{sk^*}_Y)}}. \tag{26}$$

Table 2 reports both asymptotic and bootstrap p-values, denoted as $p\text{-}v_a$ and $p\text{-}v_b$, respectively. At daily frequency, the results show that under asymptotic p-values the standard test rejects symmetry for all series, while the Bowley test rejects only for both Kerosene and Natural gas, and the Yule test only for WTI. This is expected from the results of a comparison analysis of the power functions of the previous asymptotic tests, which is provided later on. For weekly frequency and under asymptotic p-values, the standard test rejects symmetry for all series apart from Heating oil, while the Bowley test rejects for WTI, Kerosene and Natural gas, and the Yule test for Brent oil, Gasoline and Kerosene. For weekly returns and bootstrap p-values, the null is rejected only for WTI and Kerosene (Bowley and Yule tests) and Brent oil (Yule test). We can see that, using the robust tests, the null hypothesis of skewness equal to zero is never rejected for Heating oil, indicating that there is no evidence of skewed unconditional distributions for this series of returns under both frequencies. However, results differ for the rest of the series depending on the test used and also on the frequency. For instance, the daily returns series of Brent oil and Gasoline show no evidence of skewness under the two robust tests. It is also worth observing from the comparative of tests power functions provided below that if the series is not symmetric, the Yule test is expected to reject more often that the Bowley test.

4.2.2. Tests under the TGARCH-GC model

Table 3 presents maximum likelihood estimates (MLEs) of the TGARCH-GC model for the returns series in Table 1. All series at both frequencies present high persistence and kurtosis as well as an asymmetric response of volatility to positive and negative shocks, as we can see, in most cases, from the different estimates of α^+ and α^-. Note that the asymmetric volatility behavior in most series, and also under both frequencies, is in line with the empirical evidence of that the estimated values of α^+ are smaller than those of α^-, except for Natural gas daily and weekly returns. Regarding the skewness-related parameter, sk_z, it is found to be negative and statistically significant for WTI, Brent oil and Gasoline daily and weekly returns.[12] Table 4 contains the implied skewness, sk_r, obtained by plugging the TGARCH-GC parameter estimates into equation (15). Comparing these estimates with the sample skewness displayed in Table 1, we can see that both sample and implied estimates are negative for most of the series, although there are a couple of exceptions. First, the case of the Natural gas weekly returns, for which the implied skewness is positive. Second, the weekly series of the Heating oil returns, for which the sample skewness is positive but the implied one is negative.

Next, we compute 95% CIs for sk_r following the methodology described in Section 2.2. Table 4 shows the CIs calculated for all the series taking $q = 10$. The reason for such a number is because the larger q, the higher the power function of the test, as shown in Carnero et al. (2022). We can see that using this test, the null hypothesis of zero skewness is rejected only for the daily series of WTI, Brent oil and Gasoline.[13]

Finally, we perform a bootstrap test using the TGARCH-GC skewness sk_r in (15). Specifically, following the same procedure as in (26), we get 500 block-bootstrap time series, $r^*{}_t$, from the original series, r_t. For each of the bootstrap series $\{r^*{}_{t,i}\}$ where $i = 1, \ldots, 500$, we have found the estimator of the implied skewness, $\widehat{sk^*}_{r,i}$, given by the fitted TGARCH-GC model, and then we calculated the sample variance of the bootstrap implied skewness estimates by using the 500 values previously obtained, i.e. $\widehat{Var}(\widehat{sk^*}_r) = \frac{1}{499}\sum_{i=1}^{500}(\widehat{sk^*}_{r,i} - \overline{\widehat{sk^*}_r})^2$ where $\overline{\widehat{sk^*}_r} = \frac{1}{500}\sum_{i=1}^{500}\widehat{sk^*}_{r,i}$. Hence, the test statistic is given by

$$z^*{}_{\text{implied}} = \frac{\widehat{sk}_r}{\sqrt{\widehat{Var}(\widehat{sk^*}_r)}}, \tag{27}$$

and the p-value given by the standard normal distribution. The results, exhibited in Table 4, show that, returns' skewness is statistically

[11] In particular, we have used Matlab functions *fitdist. m* and *quantile. m* in order to estimate the density function f and the first and third quartiles.

[12] As a robustness check, we have also considered the TGARCH model assuming that innovations z_t in (10) follow the well-known skewed Student's t distribution of Hansen (1994). The estimation results, available upon request, for the skewness-related parameter are in line with the results obtained for the TGARCH-GC.

[13] Weekly CI results are not reported as they are not reliable enough due to small partitioned sub-sample sizes. However, they are available upon request.

Table 3
This table presents maximum likelihood estimates of the TGARCH-GC parameters for the return series in Table 1. Heteroscedasticity-consistent standard errors are provided in parentheses below the parameter estimates.

	WTI	Brent oil	Heating oil	Gasoline	Kerosene	Natural gas
Panel 1: Daily frequency						
μ	− 0.001	0.013	0.043 * **	0.028	0.028	0.066
	(0.009)	(0.007)	(0.007)	(0.012)	(0.008)	(0.033)
ω	0.030 * **	0.029 * **	0.0363 * **	0.051 * **	0.024 * **	0.121 * **
	(0.012)	(0.010)	(0.010)	(0.0013)	(0.012)	(0.019)
β	0.912 * **	0.915 * **	0.900 * **	0.897 * **	0.918 * **	0.848 * **
	(0.017)	(0.010)	(0.014)	(0.015)	(0.018)	(0.031)
α^+	0.085 * **	0.080 * **	0.119 * **	0.108 * **	0.091 * **	0.209 * **
	(0.015)	(0.014)	(0.011)	(0.013)	(0.012)	(0.018)
α^-	0.125 * **	0.116 * **	0.109 * **	0.113 * **	0.107 * **	0.150 * **
	(0.022)	(0.020)	(0.020)	(0.024)	(0.024)	(0.044)
sk_z	− 0.202 * **	− 0.122 * *	− 0.017	− 0.106 * **	− 0.052	0.062
	(0.038)	(0.037)	(0.037)	(0.035)	(0.039)	(0.048)
ek_z	1.139 * **	1.091 * *	1.027 * **	0.971 * **	0.980 * **	1.473 * **
	(0.101)	(0.086)	(0.087)	(0.076)	(0.121)	(0.145)
Panel 2: Weekly frequency						
μ	− 0.048	0.009	0.120	0.043	0.101	0.272
	(0.103)	(0.086)	(0.093)	(0.102)	(0.093)	(0.218)
ω	0.186 * **	0.158 * **	0.255 * **	0.343 * **	0.227 * **	1.019 * **
	(0.066)	(0.068)	(0.066)	(0.099)	(0.057)	(0.424)
β	0.827 * **	0.854 * **	0.805 * **	0.804 * **	0.819 * **	0.649 * **
	(0.023)	(0.035)	(0.027)	(0.035)	(0.024)	(0.087)
α^+	0.111 * **	0.106 * **	0.178 * **	0.158 * **	0.158 * **	0.437 * **
	(0.025)	(0.025)	(0.032)	(0.040)	(0.023)	(0.119)
α^-	0.259 * **	0.182 * **	0.184 * **	0.175 * **	0.180 * **	0.242 * **
	(0.063)	(0.044)	(0.030)	(0.036)	(0.031)	(0.070)
sk_z	− 0.245 * **	− 0.239 * *	− 0.046	− 0.232 * **	− 0.068	0.176
	(0.082)	(0.075)	(0.080)	(0.078)	(0.082)	(0.135)
ek_z	0.936 * **	0.555 * *	0.735 * **	0.806 * **	0.749 * **	1.528 * **
	(0.277)	(0.185)	(0.202)	(0.214)	(0.184)	(0.517)

Finally, (***) indicates significance at 1% level; (**) at 5% level and (*) at 10% level. The sample sizes for each series are provided in Table 1.

Table 4
This table presents the implied skewness of returns under the TGARCH-GC model with parameter estimates presented in Table 3; bootstrap p-values for the z-statistic in (27), as well as confidence intervals for sk_r with $q = 10$.

	WTI	Brent oil	Heating oil	Gasoline	Kerosene	Natural gas
Panel 1: Daily frequency						
implied skewness	− 0.46***	− 0.20***	− 0.03	− 0.13***	− 0.11	− 0.51
bootstrap p-value	0.00	0.00	0.61	0.00	0.08	0.05
CI	(− 0.58, − 0.12)	(− 0.39, − 0.03)	(− 0.19, 0.04)	(− 0.27, − 0.01)	(− 0.24, 0.05)	(− 0.74, 2.30)
Panel 2: Weekly frequency						
implied skewness	− 0.78	− 0.33***	− 0.06	− 0.27***	− 0.09	0.67
bootstrap p-value	0.65	0.00	0.57	0.00	0.44	0.99

Finally, (***) indicates significance at 1% level taking into account bootstrap p-values.

different from zero for WTI, Brent oil and Gasoline daily series as well as for Brent oil and Gasoline weekly series.

In order to check the robustness of our results, we have estimated the unconditional skewness of the daily returns series by means of the implied skewness given by the fitted TGARCH-GC model using a rolling window scheme, with sample size $T = 5000$. As an example, we present the results for two series of daily returns.[14] Specifically, Heating oil, for which the null hypothesis of skewness equal to zero is not rejected, and Gasoline, for which the null hypothesis of skewness equal to zero is rejected. We first estimate the TGARCH-GC model and the implied skewness using the period from June 2, 1986 to April 19, 2006. When a new observation is added to the sample, we delete the first observation and re-estimate the TGARCH-GC model and the implied skewness. This process is repeated until we reach the last 5000 observations in the sample, from May 1, 2001 to April 5, 2021. This amounts to considering 3758 different subsamples and therefore the same number of unconditional skewness estimates, which are exhibited in Fig. 3. This figure also plots the 95% IM confidence intervals for sk_r, together with the implied skewness obtained with the full sample, both displayed in Table 4. Finally, Fig. 3 also exhibits the zero line in order to see whether or not it is included in the confidence interval for sk_r. As we can see, on the one hand, for the Heating oil, the estimated skewness obtained using the 3758 different subsamples is close to zero, rather similar to the unconditional skewness obtained with the full sample, and it is always inside the 95% confidence interval. Notice that the previous CI does contain the zero line, confirming that the unconditional skewness of the Heating oil daily returns is equal to zero. On the other hand, for the Gasoline series, the estimated skewness corresponding to the different subsamples is always negative, very close to the unconditional skewness computed with the full sample, and it is always inside the 95% confidence interval, which does not contain the zero line. This result shows evidence that the unconditional distribution of the Gasoline daily returns is negatively skewed.

[14] This robustness analysis results for the rest of series in the paper are not presented to save space and as they consistent to those exhibited but they are available upon request.

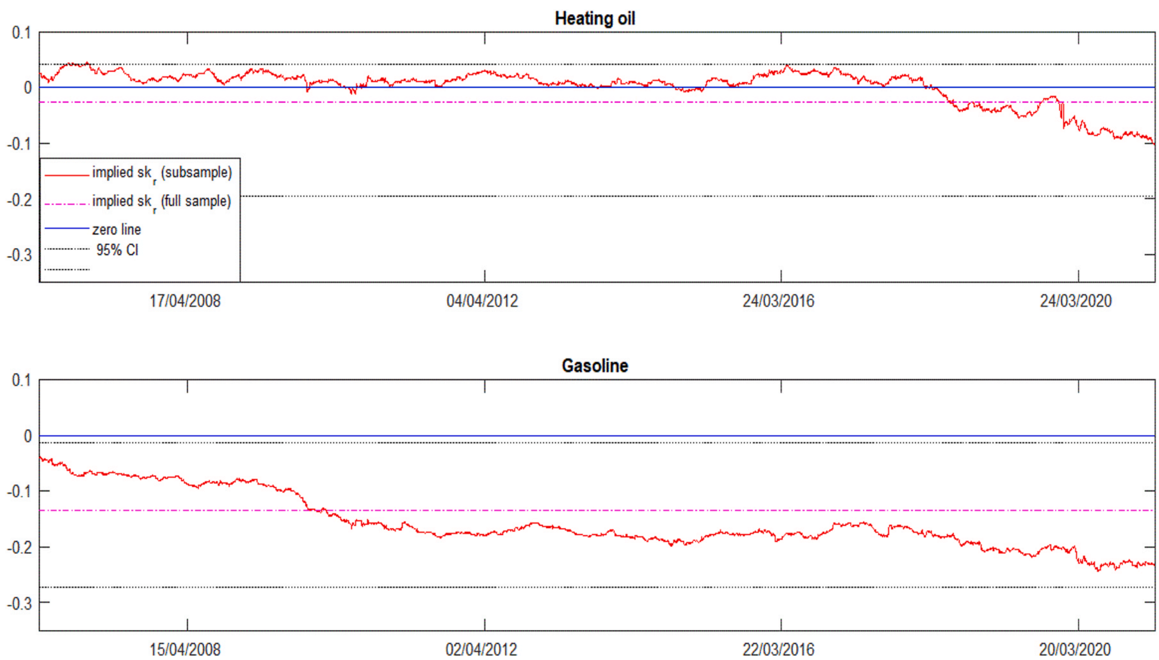

Fig. 3. TGARCH-GC implied unconditional skewness of the daily returns of Heating oil (top panel) and Gasoline (bottom panel) computed with subsamples of size $T = 5000$ using a rolling window of both series of daily returns. The dates in the x-axis refer to the end-of-window dates.

4.2.3. Skewness tests power comparisons

We provide a simulation exercise to compare the power of traditional, robust and TGARCH-GC implied skewness tests for skewness. The sample, Boyle and Yule asymptotic and bootstrap skewness test statistics are given in (9) and (26), respectively, and the TGARCH-GC implied skewness bootstrap test statistic is in (27).

We obtain 1000 replicates of TGARCH-GC series, of size $T = 3000$, given by model (10) for asset returns typical parameter values: $\mu_t = 0$, $\omega = 0.0217$, $\alpha^+ = 0.0244$, $\alpha^- = 0.1252$, $\beta = 0.9239$, $ek_z = 0.7533$ and $sk_z = 0$ (under the null hypothesis, H_0: $sk_z = 0$) and $sk_z = \pm$ 0.1, 0.2, 0.3, 0.4, 0.5, 0.6 (under H_1: $sk_z \neq 0$). The tests significance level is $\alpha = 0.05$.[15] Fig. 4 displays the estimated power function for the asymptotic (left plot) and bootstrap (right plot) skewness tests. Sample, Boyle and Yule power functions for asymptotic (bootstrap), in the left (right) plot, are denoted by Astand (Bstand), Aboyle (Bboyle) and Ayule (Byule), respectively, and the TGARCH-GC power function is denoted by BTGARCH.

Regarding the asymptotic tests, the estimated size of the test (i.e. the probability of rejecting the null hypothesis when it is true) is 0.6340 for Astand, 0.0300 for Aboyle and 0.0440 for Ayule. Clearly, Astand performs much worse than the asymptotic robust tests in regard to size, the latter provide rather close values to the nominal level, not only for $\alpha = 0.05$, but also for $\alpha = 0.01, 0.10$ -these last results are available upon request. This illustrates the high probability of rejecting symmetric distributions when employing sample asymptotic tests. Under H_1, the Bowley asymptotic test estimated power function is much worse than that of Yule and sample tests, needing very high levels of skewness to be able to reject the null successfully. Turning to the bootstrap tests, we find that all provide very similar and close size to the nominal level. TGARCH estimated power is much better, as expected, than the rest of the tests. It is confirmed that the Bowley bootstrap test performs much worse too than Yule and sample ones. Finally, Yule test estimated power is better than that of the sample test.

4.3. VaR and ES accuracy

In order to evaluate sample and implied estimated VaR and ES relative accuracy, we consider empirical number of violations as well as magnitude of exceptions, see López (1998) and Angelidis et al. (2007). For this purpose, we implement a constant-size rolling window exercise with $N_d = 2000$ and $N_w = 500$ OOS estimations for daily and weekly frequencies, respectively.

Let $h_t(\alpha) = \mathscr{I}(r_t < X)$ be the violation or hit variable with $X = VaR(\alpha)$, $ES(\alpha)$ as the unconditional VaR and ES measures in section 3.2. Let $\widehat{h_t(\alpha)}$ be the estimation of $h_t(\alpha)$, i.e. $\widehat{h_t(\alpha)} = \mathscr{I}(r_t < \widehat{X_t})$ where $\widehat{X_t} = \{\widehat{VaRj_t(\alpha)}, \widehat{ESj_t(\alpha)}: j = 2, 3\}$. Note that both $\widehat{VaRj_t(\alpha)}$ and $\widehat{ESj_t(\alpha)}$ depend now on the subscript t since they are obtained with the estimation of the parameters under the information set at time t determined by the rolling window procedure. The quadratic loss function, which incorporates the exception magnitude, is $QL_t(\alpha) = (r_t - X)^2 \times h_t(\alpha)$. The sample average of the estimation of $QL_t(\alpha)$ over the OOS period is given by $\widehat{AQL(\alpha)} = N^{-1}\sum_{t=1}^{N}\widehat{QL_t(\alpha)}$, where $\widehat{QL_t(\alpha)} = (r_t - \widehat{X_t})^2 \times \widehat{h_t(\alpha)}$ and $N = N_d, N_w$. The risk measures of $\widehat{X_t}$ are obtained under two alternative estimation methods. First, the 'implied method' based on estimating the unconditional moments μ, σ_r^2 and sk_r as a function on the ML parameter estimates from the TGARCH-GC model described in section 2.2.1. Second, the 'sample method' that consists on replacing the above moments by their sample ones.

Table 5 reports mean values across the OOS period corresponding to measures of $\widehat{X_t}$ for comparison purposes.[16] Our results show that, in most cases, the mean values of $\widehat{VaR2_t(\alpha)}$ and $\widehat{ES2_t(\alpha)}$ are lower (in absolute value) than $\widehat{VaR3_t(\alpha)}$ and $\widehat{ES3_t(\alpha)}$ for each method and frequency. Indeed, we find that the latter are more conservative measures of tail risk (higher absolute values) than the former when the average (sample and implied) skewness over the N windows of

[15] An analysis of the test power in relation to lower sample sizes, e.g. $T < 1000$ typical for weekly data, is discarded in our simulation exercise because of the same reason explained in footnote 13.

[16] Those cases where the CF3 quantile estimation is non-monotonic, i.e. $\varphi'(w_\alpha) < 0$, the *increasing rearrangement* procedure by Chernozhukov et al. (2010) has been implemented to restore monotonicity. An empirical application of this methodology can be seen in Amédée-Manesme et al. (2015).

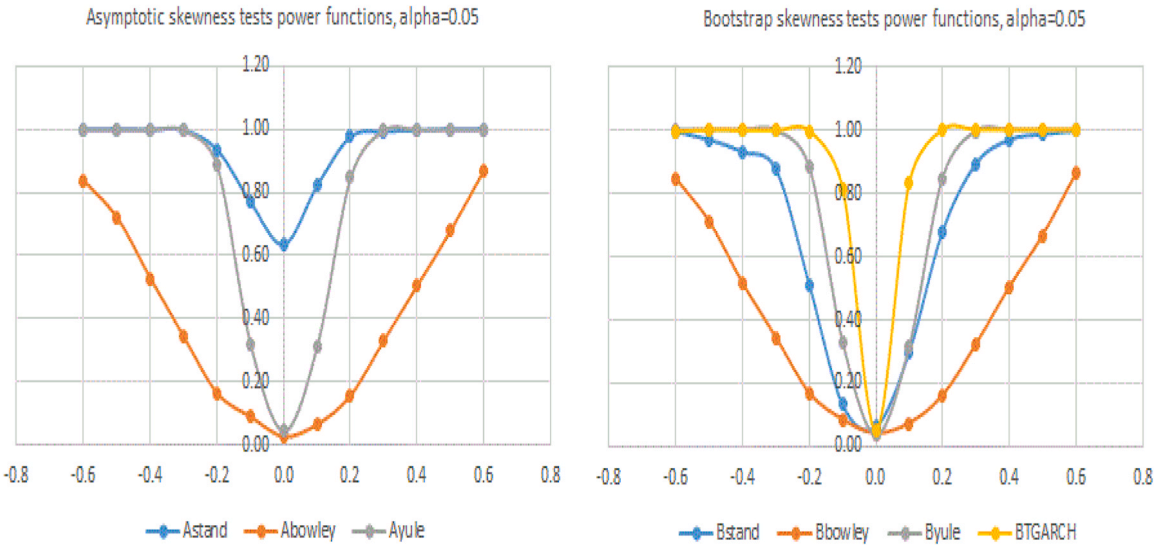

Fig. 4. The left panel exhibits plots for the power of sample, Bowley and Yule asymptotic skewness tests, denoted as Astand, Abowley and Ayule, respectively, whose statistics are given in (9). The right panel displays the power functions for sample, Bowley, Yule and TGARCH-GC-implied bootstrap skewness tests, denoted as Bstand, Bbowley, Byule and BTGARCH, whose statistics are given in (26) and (27). For all cases the significance level is alpha = 0.05. The results are based on 1000 replicates of TGARCH-GC series, of sample size $T = 3000$, given by model (10) taking asset returns tipical parameter values: $\mu_t = 0$, $\omega = 0.0217$, $\alpha^+ = 0.0244$, $\alpha^- = 0.1252$, $\beta = 0.9239$, $ek_z = 0.7533$ and $sk_z = 0$ (under H_0) and $sk_z = \pm$ 0.1, 0.2, 0.3, 0.4, 0.5 and 0.6 (under H_1).

Table 5
This table provides average one per cent ($\alpha = 0.01$) third-order Cornish-Fisher approximated VaR and ES for the OOS period from sample moments and TGARCH-GC-implied moments. Predictions: 2000 (daily), 500 (weekly).

	Sample	Implied	Sample	Implied	Sample	Implied	Sample	Implied
	Panel 1: Daily frequency				Panel 2: Weekly frequency			
	WTI		Brent oil		WTI		Brent oil	
VaR2	− 5.69	− 9.10	− 5.23	− 7.95	− 10.03	− 10.43	− 9.76	− 11.34
ES2	− 6.52	− 10.42	− 5.99	− 9.11	− 11.51	− 11.96	− 11.20	− 12.99
VaR3	− 6.67	− 10.16	− 6.17	− 8.54	− 11.55	− 11.41	− 10.82	− 12.35
ES3	− 7.90	− 11.91	− 7.31	− 9.94	− 13.64	− 13.33	− 12.69	− 14.41
	Heating oil		Gasoline		Heating oil		Gasoline	
VaR2	− 5.47	− 6.89	− 6.13	− 6.82	− 10.10	− 11.34	− 11.15	− 11.87
ES2	− 6.26	− 7.90	− 7.02	− 7.82	− 11.59	− 13.01	− 12.79	− 13.60
VaR3	− 8.02	− 6.90	− 6.53	− 7.15	− 9.74	− 11.44	− 11.26	− 12.98
ES3	− 9.85	− 7.92	− 7.59	− 8.29	− 11.09	− 13.15	− 12.94	− 15.16
	Kerosene		Natural gas		Kerosene		Natural gas	
VaR2	− 5.81	− 10.23	− 10.34	− 19.12	− 10.04	− 10.70	− 17.37	− 20.21
ES2	− 6.65	− 11.73	− 11.84	− 21.91	− 11.52	− 12.28	− 19.89	− 23.15
VaR3	− 6.01	− 10.64	− 8.36	− 17.98	− 10.61	− 11.00	− 15.07	− 19.65
ES3	− 6.94	− 12.30	− 9.06	− 20.44	− 12.32	− 12.71	− 16.66	− 22.37

the OOS period is negative. Only for Heating oil (weekly) and Natural gas (daily and weekly), $\widehat{VaR3_t(\alpha)}$ and $\widehat{ES3_t(\alpha)}$ under the sample method are lower (in absolute value) than their corresponding $\widehat{VaR2_t(\alpha)}$ and $\widehat{ES2_t(\alpha)}$ since for these series the OOS average sample skewness is positive. The same pattern occurs for Natural gas (daily and weekly) under the implied method, as the OOS average implied skewness is positive for these series.

Table 6 shows the number of violations $\sum_{t=1}^{N}\widehat{h_t(\alpha)}$ and the average of the magnitude of exceptions (AQL). We use these measures to evaluate the accuracy of two-moment risk measures (VaR2, ES2) versus three-moment ones (VaR3, ES3), and sample versus implied estimated VaR and ES. Several conclusions are obtained. First, the VaR results (Panel 1) show that in most cases the number of exceptions from $\widehat{VaR3_t(\alpha)}$ under both methods are no higher than their respective ones from $\widehat{VaR2_t(\alpha)}$, as well as the former exhibit lower AQL values than the latter, for both frequencies. As a peculiarity, we find that (i) for Heating oil (weekly), $\widehat{VaR3_t(\alpha)}$ presents a higher number of violations (and higher AQL) than $\widehat{VaR2_t(\alpha)}$ under the sample method; and (ii) for Natural gas (daily and weekly), $\widehat{VaR3_t(\alpha)}$ present a higher number of violations (and higher AQL) than $\widehat{VaR2_t(\alpha)}$ under both methods. This is because for these cases the sample and/or implied OOS average skewness are positive, in addition Natural gas daily returns' skewness is not statistically significantly different from zero as it can be seen from the implied skewness test outcomes exhibited in Table 4. Similar conclusions can be inferred from the ES results in Panel 2.[17] Second, both two-moment and three-moment VaR and ES measures from the implied method provide lower AQL values than their sample counterparts with the following exceptions: Heating oil (daily) and WTI (weekly) series for VaR3 and ES3.

In summary, our results suggest that VaR3 and ES3 significantly improve VaR2 and ES2 accuracy as the former measures deliver closer number of exceptions to the expected ones as well as lower

[17] The expected number of violations has been adjusted for the OOS sample. Specifically, several days (or weeks) have been eliminated since the TGARCH-GC parameter estimations do not verify the condition of $a_3 < 1$, see equation (17). The failure to comply with this condition means the non-existence of the third moment. As a result, the expected one-percent VaR number of violations for each series, exhibited in brackets next to series names in Table 6, is different for those series requiring the previous adjustment.

Table 6
This table shows the VaR and ES violations and corresponding *AQL* function (in parenthesis), respectively, for the one per cent ($\alpha = 0.01$) third-order Cornish-Fisher approximated VaR and ES from sample moments, and TGARCH-GC-implied moments. The expected number of violations, in brackets next to the series, has been adjusted to the number of OOS days for which condition $a_3 < 1$ in eq. (17) is verified. Predictions: 2000 (daily), 500 (weekly).

	Sample Daily frequency	Implied	Sample	Implied	Sample Weekly frequency	Implied	Sample	Implied
Panel 1: Value-at-Risk								
	WTI [20]		Brent oil [20]		WTI [4.8]		Brent oil [5]	
VaR2	44 (2.111)	23 (1.754)	39 (2.597)	16 (2.173)	8 (62.493)	7 (61.772)	8 (4.205)	6 (3.351)
VaR3	32 (1.997)	17 (1.639)	27 (2.492)	14 (2.089)	7 (60.289)	6 (60.809)	7 (3.431)	6 (2.939)
	Heating oil [20]		Gasoline [20]		Heating oil [5]		Gasoline [5]	
VaR2	30 (0.361)	18 (0.229)	37 (1.132)	25 (1.017)	8 (0.828)	4 (0.579)	3 (4.114)	3 (3.730)
VaR3	13 (0.188)	18 (0.226)	29 (1.060)	24 (0.964)	9 (0.855)	3 (0.550)	3 (3.830)	3 (3.335)
	Kerosene [19]		Natural gas [14]		Kerosene [5]		Natural gas [4.9]	
VaR2	15 (0.178)	2 (0.047)	15 (1.877)	6 (0.509)	5 (2.079)	5 (1.695)	4 (0.434)	2 (0.266)
VaR3	13 (0.176)	2 (0.041)	31 (2.378)	6 (0.753)	5 (1.857)	4 (1.604)	9 (0.748)	2 (0.292)
Panel 2: Expected shortfall								
	WTI		Brent oil		WTI		Brent oil	
ES2	33 (1.928)	14 (1.577)	27 (2.445)	12 (2.010)	6 (61.072)	6 (60.273)	8 (3.659)	6 (2.803)
ES3	23 (1.794)	11 (1.435)	19 (2.324)	11 (1.904)	6 (58.088)	6 (58.974)	5 (2.718)	4 (2.310)
	Heating oil		Gasoline		Heating oil		Gasoline	
ES2	25 (0.287)	12 (0.171)	24 (1.000)	20 (0.888)	4 (0.631)	3 (0.409)	3 (3.634)	3 (3.229)
ES3	7 (0.116)	11 (0.168)	18 (0.918)	15 (0.826)	5 (0.660)	3 (0.376)	3 (3.277)	3 (2.725)
	Kerosene		Natural gas		Kerosene		Natural gas	
ES2	13 (0.141)	1 (0.032)	11 (1.621)	4 (0.331)	4 (1.749)	4 (1.355)	2 (0.261)	1 (0.136)
ES3	13 (0.140)	1 (0.025)	24 (2.250)	4 (0.615)	4 (1.474)	4 (1.244)	5 (0.583)	2 (0.157)

AQL for both sample and implied methods and frequencies. Furthermore, AQL values (number of exceptions) from VaR2 and ES2 are higher (no lower) than those obtained under VaR3 and ES3, which means that the former measures systematically underestimate tail risk as a result of skewness not being accounted for.

5. Conclusions

In this paper we analyze whether or not energy returns have a skewed unconditional distribution. To address the problem, we first use traditional tests of skewness which most of them agree in pointing out a distribution with zero skewness for Heating oil return series. Then, we use the TGARCH model assuming a Gram-Charlier distribution for the innovations to estimate the unconditional skewness of the return series. We show that testing for skewed returns through the statistical significance of the skewness implied by the estimated TGARCH-GC modeldiffers from traditional skewness tests. Our results give evidence of negative skewness for the series of crude oil (WTI and Brent) and Gasoline returns. Finally, we show how skewness affects tail risk through an empirical application for VaR and ES. We find that three-moment VaR and ES are consistently lower than two-moment ones, which means that they provide more conservative measures of tail risk. Besides, improvements in accuracy from TGARCH-GC-implied VaR and ES respecting sample moment ones are also rather apparent. These results have important implications for risk management.

A fruitful avenue for future research would be obtaining the orthogonal decomposition of total tail risk into both skewness and kurtosis tail risk components. This kind of analysis would develop the approach in You and Daigler (2010) for VaR under the CF4 expansion. Specifically, we propose using polynomial adjusted densities as an alternative and simple method, see León and Ñíguez (2022), in order to disentangle the tail risk marginal contribution of skewness and kurtosis in the ES measure.

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Declaration of Competing Interest

We, M. Angeles Carnero, Angel Leon and Trino Ñíguez, confirm that we do not have any conflict of interest relating to this submission.

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