High-Order Hybrid Stratified Sampling: Fast Uniform-Convergence Fourier Transform Estimation

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Abstract— This paper considers the problem of estimating the Fourier transform of continuous-time signals from N nonuniformly collected observations. Here, we introduce a new class of Hybrid Stratified sampling scheme in conjunction with a suitable estimator, which can provide the rate of convergence of order $1/N^{(2K+3)}$ in the mean-square sense, for signals with $K+1$ continuous derivatives. Most importantly, it is shown that this rate is not only faster, but also uniform and independent of the analysed frequency (unlike) compared with other existing random-sampling-based techniques. In this paper, we establish the statistical properties of the proposed approach and illustrate its performance analytically as well as numerically.

Keywords—Fourier transform, random sampling, stratified sampling, convergence rate

I. INTRODUCTION

Fourier Transform (FT) calculations from samples of a real-valued signal is a fundamental task that appears in various areas of science: astronomy, seismology, biomedical sciences, NMR spectroscopy, dynamics, image analysis, optics, to name a few. This task is typically straightforward when an adequate number of samples of the processed signal can be uniformly collected at sufficiently high rates. However, the FT calculation problem becomes significantly more challenging when these sampling conditions cannot be met. Scientists, engineers and mathematicians have investigated this problem from various start points, with diverse objectives, using different approaches. Nonetheless, all these methods intrinsically aim to minimise the number of the signal samples without undermining the FT calculation accuracy in order to, for example, ease calculations, accelerate the data acquisition procedure and reduce the associated hardware requirements. One of the low-complexity approaches to FT calculation using a relatively reduced number of samples belong to a paradigm dubbed Digital Alias-Free Signal Processing (DASP) [1-10]; it entails capturing and processing *random non-equidistant* signal samples. In general, DASP approaches offer the means to mitigate the effect of the aliasing phenomenon which can have detrimental impact on the accuracy of the classical uniform-sampling FT estimation when the sampling conditions are not fulfilled. For example, this capability has been effectively utilized for wideband spectrum sensing or surveillance at notably low sampling rates [11-15].

Nevertheless, the main limitation of the FT estimation techniques based on random sampling is their slow convergence rate regardless of the smoothness of the signal [8, 9]. For instance, the basic method, known as total random (ToRa) sampling estimation [7], converges in the mean-square sense at the rate of $1/N$, where N is the number of collected samples. The development of stratified sampling (StSa) and antithetical stratified (AntSt) sampling FT estimators in [16] and [17] have shown to provide considerable potential, delivering expedited convergence rates of $1/N^3$ and $1/N^5$, respectively, for sufficiently smooth signals. However, these fast rates are *pointwise* convergence rates and are not realized as the assessed frequencies increase, unless the number of samples N is excessively large. This limitation can undermine the key benefits of DASP-based estimators, namely employing a reduced number of data samples. Additionally, the *uniform* convergence rate of these estimators, i.e. for all frequencies, is only $1/N$ [5]. In [18, 19], a class of first-order hybrid stratified (HySt) estimator that has a uniform convergence rate of $1/N^5$ was introduced. In this paper, we present a new class of hybrid stratified estimation of order K, referred to henceforth by K-HySt. Most importantly, it provides a uniform convergence rate of $1/N^{(2K+3)}$ for signals with $K+1$ continuous derivatives.

Due to space limitations, we focus here on describing the key elements of the proposed novel technique. The theories and expressions that are believed to be essential to the user for applying it are given. In Section II, the problem is formulated and the notations used throughout this paper are defined. In Section III, we briefly summarise the existing random sampling FT estimation schemes and their key characteristics. In Section IV, we describe and briefly analyse the introduced approach, establishing some of its statistical properties. A numerical example is shown in Section V to demonstrate K-HySt superior performance and illustrate the analytical observations in Sections III and IV. It is noted that proofs of all the theorems stated in this paper will be detailed in a full companion journal paper, where additional relevant statistical properties are analytically determined and numerically verified.

II. PROBLEM FORMULATION

The Fourier transform of a real-valued, finite-energy, deterministic, continuous-time signal $x(t)$ is given by:

$$X(f) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\mathrm{d}t. \quad (1)$$

The signal however can only be observed over a finite-length window $\mathcal{T}$. The objective in practice is then to estimate the following Fourier transform:

$$X_{\mathcal{T}}(f) \triangleq \int_{\mathcal{T}} x(t)e^{-j2\pi ft}\mathrm{d}t. \quad (2)$$

In classical signal processing, the integral in (2) is often approximated using a summation based on uniformly distributed samples of the signal. This approximation can severely suffer from errors induced by the spectrum aliases if the sampling rate is insufficiently high to ensure that these aliases are distance enough from one another. On the other hand, we recall that a DASP-based FT calculation from a randomly selected samples can overcome this requirement. However, DASP approaches also introduce errors, albeit of a different nature, which we quantify below in terms of the FT mean-square estimation error. Whilst the latter error typically converges to zero with increasing N values, the aim of the proposed method is to minimise it for an available set of N signal samples. We note that decreasing the number of samples is equivalent to decreasing the sampling density in view of the fact that the observation interval is fixed. Table 1 lists the notations used in this paper.

TABLE 1. NOTATIONS

$x(t)$	Analysed continuous-time signal
$X_{\mathcal{T}}(f)$	Targeted Fourier transform of $x(t)$
$\mathcal{T}$	A finite-duration observation window
H	Length of $\mathcal{T}$
N	Number of processed samples
$\hat{X}_N(f)$,	FT estimates from N samples
$\mathcal{T}_l$	The l^{th} stratum given
$\mathcal{T}_{l,m}$	The m^{th}stratum of the l^{th}subset
τ_l	Random sampling instant in $\mathcal{T}_l$
$\tau_{l,m}$	Random sampling instant in $\mathcal{T}_{l,m}$
τ_n	The n^{th} random sampling instant
L_N	Number of strata
K	Order of Hybrid Estimation
$L_{N,K}$	Number of subsets of K adjacent stratum
Δ_l	Length of $\mathcal{T}_l$
$\Delta_{l,m}$	Length of $\mathcal{T}_{l,m}$
t_l or $t_{l,m}$	Edges of strata
$\mathcal{D}_l$	The l^{th} subset of K adjacent strata
$h(t)$	Stratifying function

III. BACKGROUND: RANDOM SAMPLING FOURIER TRANSFORM ESTIMATION SCHEMES

Each of the aforementioned total random, stratified sampling, and antithetical stratified sampling FT estimation approaches comprises a sampling scheme and a corresponding estimator. To approximate the integral in (2) and obtain an estimate of $X_{\mathcal{T}}(f)$ using N signal samples, the estimators of these three techniques, in general, follow the summation of the form:

$$\hat{X}_N(f) = \sum_{n=1}^{N} x(\tau_n)\, e^{-j2\pi f\tau_n} A_n \ , \quad (3)$$

where $\{A_n\}_{n=1}^{N}$ components are constants that depend on the sampling scheme and ensure the unbiased nature of the estimator with respect to $X_{\mathcal{T}}(f)$. Starting with ToRa [7], the sampling instants are independent, identically distributed (IID) random variables with a chosen probability density function (PDF) that takes strictly positive values within the observation window and zero elsewhere. From [7], the rate of decay of the ToRa mean-square FT estimation error is of order N^{-1}.

For the StSa sampling technique, the observation interval $\mathcal{T}$ is divided into N non-overlapping subintervals, according to a PDF-like function $h(t)$. Each of the sampling instants in $\{\tau_n\}_{n=1}^{N}$ is selected randomly and independently as per a uniform PDF whose value is non-zero within the corresponding subinterval and zero elsewhere. The rate of convergence of the StSa estimates cannot be determined for a finite number of N, i.e. from the exact expression of the variance of the StSa estimator. As an alternative, it is standard in statistical estimation to determine the rate of convergence from an asymptotic expression of the variance as N tends towards infinity. An asymptotic expression of the variance of the StSa estimates was determined in [16] where it is illustrated that the mean-square FT estimation error converges to zero at the rate of N^{-3} for signals with first-order continuous derivative.

For the AntSt sampling scheme and in order to collect N samples, the window $\mathcal{T}$ is divided into $N/2$ subintervals and then two sampling instants are selected inside each subinterval. The first sampling instant is randomly chosen in the same way as in StSa scheme, whereas the second sampling point is the symmetrical reflection of the first one around the centre point of their time subinterval. Thus, each random sampling instant τ_n is accompanied by another sampling instant at $2c_n - \tau_n$, where c_n is the centre point of the n-th time subinterval. The mean-square FT estimation error for the AntSt approach asymptotically converges to zero at the rate of N^{-5} for signals with second order continuous derivatives [17].

Nonetheless, it was shown in [18, 19] that the convergence rates of the above approaches are pointwise. In fact, the uniform convergence rates of StSa and AnSt FT estimation is no faster than and exactly equal to N^{-1}. Hence, the StSa and AnSt techniques are characterized by a slow uniform convergence rates compared with their stated faster pointwise convergence rates, i.e. N^{-3} and N^{-5}. Whereas, the pointwise and uniform convergence rates for ToRa are the same (this also applies to *K*-HySt as will be seen in the next section).

The main consequence of the slow uniform convergence rate of the FT estimation errors of the two stratification based approaches StSa and AntSt is that they show distinctively different behavior for different assessed frequencies. More specifically, the estimation errors at individual frequencies converge slowly (at the rate of N^{-1}) when N is relatively small). Once N passes a certain threshold, this rate accelerates to N^{-3} and N^{-5}, respectively. These thresholds depend on the frequency at which the FT is estimated. The higher the frequency the more samples are required to initiate the faster convergence. Therefore, when analysing frequency ranges that are relatively wide compared with the sampling density, StSa and AnSt schemes show similar performance with no gains over the basic ToRa method.

Thereby, developing the proposed *K*-HySt approach is motivated by the need to construct a consistently accurate FT estimator, i.e. regardless of the considered frequency, that uses a low sampling density, e.g. when exploring wide frequency ranges. It involves no additional substantial computational cost compared with other efficient DASP-based estimators.

IV. *K*-Order Hybrid Stratified Estimation

For the introduced *K*-HySt scheme, the observation window $\mathcal{T}$ is divided into L_N non-overlapping subintervals. The partitions are defined in two steps according to the PDF-like function denoted by $h(t)$, which is separated from 0 by h_{min}. Let $\mathcal{D}_l$, $l = 1, \ldots, L_{N,K}$, be the subsets of K adjacent strata, where $L_{N,K} = L_N/K$. The margins of each subset are first defined by:

$$\int_0^{t_l} h(t)\mathrm{d}t = H\frac{l}{L_{N,K}}, \;\; l = 0, \ldots, L_{N,K}. \tag{4}$$

The margins of the strata within a subset are obtained through

$$t_{l,m} = t_l + \frac{m}{K}\Delta_l, \qquad m = 0,1, \ldots, K. \tag{5}$$

The *K*-HySt sampling points are collected at the partitions of the strata and one sampling point is randomly selected within each stratum, as per a uniform PDF. Subsequently, the number of samples are given by: $N = 2L_{N,k}K + 1$. To avoid introducing new notation, the stratum $[t_{l,m-1}, t_{l,m}]$ is denoted by $\mathcal{T}_{l,m}$ with length $\Delta_{l,m} = t_{l,m-1} - t_{l,m}$, and the randomly selected time instant within by $\tau_{l,m}$. For a more compact presentation of the expressions, let

$$\lambda(t,f) = e^{-j2\pi f t} \, . \tag{6}$$

The *K*-HySt estimator is consequently given by

$$\hat{X}_N(f) = \sum_{l=1}^{L_{N,K}} \sum_{m=1}^{K} \hat{I}_{l,m}(f)\,, \tag{7}$$

where

$\hat{I}_{l,m}(f) =$

$$\Delta_{l,m} x(\tau_{l,m})\lambda(\tau_{l,m},f) + \textstyle\sum_{k=0}^{K} x(t_{l,k}) \int_{\mathcal{T}_{l,m}} \gamma_{l,k}(t)\lambda(t,f)dt$$

$$- \textstyle\sum_{k=0}^{K} \Delta_{l,m} x(t_{l,k})\, \gamma_{l,k}(\tau_{l,m})\, \lambda(\tau_{l,m},f), \tag{8}$$

and

$$\gamma_{l,k}(t) \mathrel{\widehat{=}} \prod_{\substack{0\le g\le K \\ g\ne k}} \frac{t - t_{l,g}}{t_{l,k} - t_{l,g}}. \tag{9}$$

This can be shown to lead the following three key theorems:

Theorem 1: The *K*-HySt estimator (7) is an unbiased estimator of the windowed FT in (2):

$$E\{\hat{X}_N(f)\} = X_{\mathcal{T}}(f). \tag{10}$$

Theorem 2: If the signal $x(t)$ has continuous $K+1$ derivatives, the *K*-HySt estimator converges uniformly to $X_{\mathcal{T}}(f)$ at least at the rate of $N^{-(2K+3)}$:

$$\mathrm{Var}\{\hat{X}_N(f)\} < 0.5N^{-(2K+3)}\left(\left[C_K + \frac{K!\,K}{4}\right]\left[\frac{3H}{h_{min}}\right]^{K+2} \frac{x_{k+1,max}}{(K+1)!}\right)^2,$$

where $x_{K+1,max} \mathrel{\widehat{=}} \sup_{t\in\mathcal{T}} \left|x^{(K+1)}(t)\right|$, $x^{(k)}(t)$ denotes the k^{th} derivatives of $x(t)$, and $C_K = \sum_{m=1}^{K}\left|C_{K,m}\right|$ is a constant that can be obtained from Table 2 for a few K values.

Theorem 2 demonstrates that the *K*-HySt estimator provides a frequency-independent upper bound on the Fourier transform estimation errors that decays to zero at the rate $N^{-(2K+3)}$, or faster. The achieved accelerated convergence transpires simultaneously across all frequencies, unlike StSa and AnSt.

Theorem 3: Assume that the function $x(t)$ has $K+1$ continuous derivatives then

$$\lim_{N\to\infty} N^{2K+3}\mathrm{Var}\{\hat{X}_N(f)\} = \sigma^2_{K-HySt,lim}\,, \tag{11}$$

where,

$$\sigma^2_{K-HySt,lim} \mathrel{\widehat{=}} \frac{D_K(2H)^{2K+3}}{[(K+1)!]^2 K^{2K+4}} \int_{\mathcal{T}} \frac{\left[x^{(K+1)}(t)\right]^2}{h^{2K+3}(t)} dt. \tag{12}$$

where,

$D_K = \sum_{m=1}^{K} D_{K,m}$ is a constant that depends only on the order K, where $D_{K,m} \mathrel{\widehat{=}} G_{K,m} - C^2_{K,m}$. The values of the constants $G_{K,m}$ can be obtained from Table 2 for a few K values.

TABLE 2. $C_{K,m}$ AND $G_{K,m}$ CONSTANTS

$K = 1$	$C_{1,1} = -1/6$	$G_{1,1} = 1/30$
$K = 2$	$C_{2,1} = -C_{2,2} = 1/4$	$G_{1,2} = G_{2,2} = 8/105$
$K = 3$	$C_{3,1} = C_{3,3} = -19/30$ $C_{3,2} = 11/30$	$G_{3,1} = G_{3,3} = 313/630$ $G_{3,2} = 103/630$
$K = 4$	$C_{4,1} = -C_{4,4} = 9/4$ $C_{4,3} = -C_{4,2} = 11/12$	$G_{4,1} = G_{4,4} = 4408/693$ $G_{4,2} = G_{4,3} = 712/693$

In summary, the above results have the following implications:

1- Estimation quality is measured by the mean square error:
$$Err_N(f) \triangleq \mathrm{E}\left\{\left|\hat{X}_N(f) - X_{\mathcal{T}}(f)\right|^2\right\}$$
where $\hat{X}_N(f)$ denotes the FT estimates constructed from N samples of the signal $x(t)$. For unbiased estimators, $Err_N(f)$ and the variance $\mathrm{Var}\{\hat{X}_N(f)\}$ are identical.

2- The rate of convergence of K-HySt is exactly $1/N^{2K+3}$, for signals with $K+1$ continuous derivatives as per
$$\lim_{N\to\infty} N^{2K+3}\,\mathrm{E}\left\{\left|\hat{X}_N(f) - X_{\mathcal{T}}(f)\right|^2\right\} = \sigma^2_{K-HySt,lim}$$

3- Theorem 3 shows that the asymptotic constant is not a function of the analysed frequency f, as opposed to the asymptotic constants of the StSa and AntSt estimators in [16], [17]. The asymptotic expression in equation (12) can help practitioners to anticipate (determine in advance) the estimation error and decide accordingly on K and N.

4- The relation between the number of collected samples N and the FT estimation error for K-HySt is hardly affected by the analysed frequency at which the FT is estimated.

V. Numerical Analysis

A numerical example is presented in this section to illustrate the superior performance of the K-HySt FT estimator, compared with the other DASP approaches, ToRa, StSa and AnSt. We consider a semi-synthetic signal, which constitutes a large number of sinusoidal components. The magnitude of its Fourier transform is depicted in Fig. 1 for the entire frequency range, Fig. 1(a), as well as around two particular components of interest, Fig. 1(b). This signal mimics accelerations measured by an accelerometer mounted to a wave sensor. The objective here is to apply a random sampling Fourier transform estimation to explore wide frequency ranges beyond the Nyquist sampling rate, in order to identify high frequency harmonics. The Fourier transform in the targeted high frequency range is highlighted in Fig. 1(b). The processed signal is observed over a window of length 10 seconds. For each of the three stratification-based techniques, we use $h(t) = 1$. This implies that all of the strata are of the same length. For ToRa, a uniform PDF across the observation window is used to generate the IID random sampling instants.

The tackled challenging task in this example is to estimate FT for a wide frequency range using a low number of samples $N = 80$. With an observation window of length 10 seconds, the sampling density is 8 Hz. Such rate, if uniform sampling is employed, would only permit exploring a frequency range up to 4 Hz as per the Nyqyist criterion. To display the convergence rates of the FT estimates, we plot the mean-square estimation errors at selected frequency points versus the number of collected random samples. The mean-square estimation error is obtained by averaging the errors obtained from 5,000 independent, Monte Carlo, simulations.

In Figs. 2 and 3, we respectively show the mean-square error for ToRa, StSa, AnSt and 2-HySt methods against N at frequencies 170 Hz and 402 Hz, where the harmonics in Fig. 1(b) are located. They reveal that the fast convergence rates of the StSa and AntSa are not evident for this range of frequencies. Both estimators do not only exhibit similar performance, but also their accuracy is very close to that of the basic ToRa approach in [7]. The accelerated convergence rates of StSa and AntSa would only become visible when the number of collected samples N is sufficiently large for the considered frequencies, in particular $N \approx 10{,}000$ for this case. Whereas, the proposed 2-HySt scheme delivers a notable improvement in performance compared with all the above DASP estimators. This clearly demonstrates that K-HySt scheme is not as frequency-sensitive as its stratification-based predecessors. To depict the acceleration in the convergence rates of StSa and AntSa, we are required to reduce the analysed frequency to as low as 3 Hz. The mean-square estimation error for this frequency is presented in Fig. 4. Overall, the numerical results in this section are consistent with the analytical results of Sections III and IV.

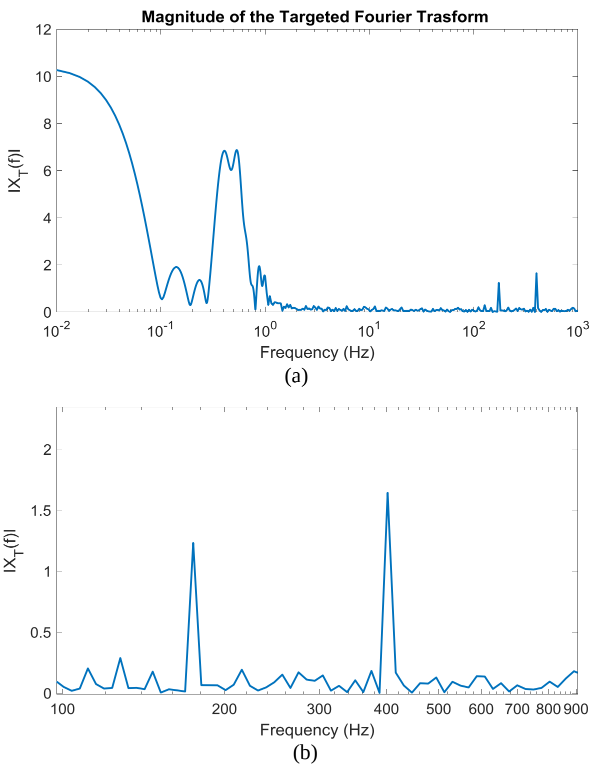

Fig. 1. Magnitude of the Fourier Transform of the considered signal. (a) Over the signal whole frequency range. (b) For a selected frequency range around the two sinusoidal components located at 170 Hz and 402 Hz.

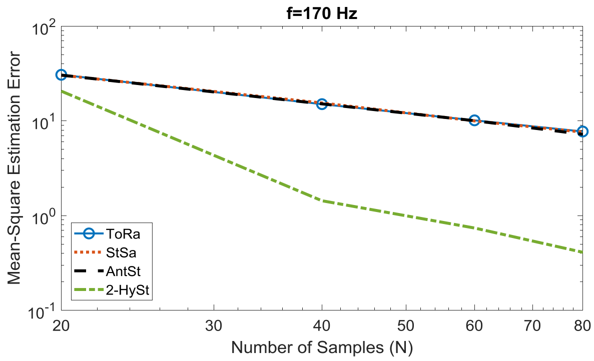

Fig. 2. The mean-square estimation error at frequency point 170 Hz versus the number of samples.

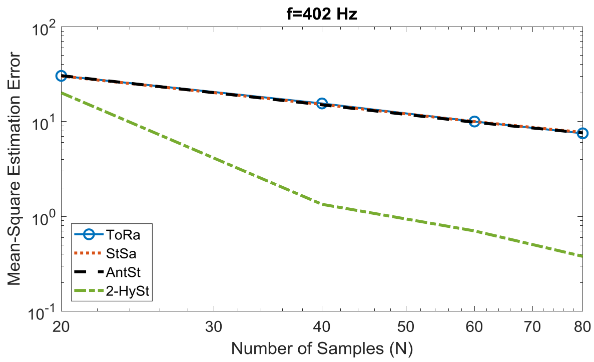

Fig. 3. The mean-square estimation error at frequency point 402 Hz versus the number of samples.

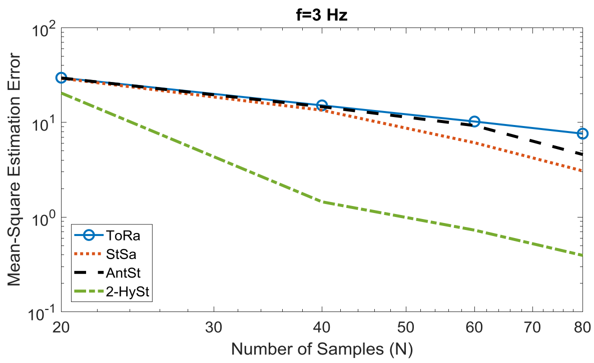

Fig. 4. The mean-square estimation error versus the number of samples at frequency 3 Hz, where the standard stratification techniques exhibit an acceleration in their convergence rates.

VI. Conclusions

A generalisation of the hybrid-stratified-sampling method, i.e. K-HySt, to estimate the FT of a deterministic continuous-time signal, at arbitrary frequencies, from a number of its samples is introduced. Its mean-square error can be shown to uniformly converge to zero at a rate of $1/N^{2K+3}$, enabling it to deliver more accurate estimates compared with its alias-free-type predecessors. This paper serves to motivate further research into using the simple and low complexity alias-free sampling approach to calculate the signal Fourier transform form remarkably low number of samples.

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