Socratic Proofs for Propositional Linear-Time Logic

Urbanski, M., Bolotov, A., Shangin, V. and Grigoriev, O.

This is an electronic version of a paper presented at *the Joint Automated Reasoning Workshop and Deduktionstreffen,* Vienna 23 to 24th July 2014.

The conference proceedings are available online at:

http://arw.csc.liv.ac.uk/year/2014/ARW2014proceedings.pdf

Socratic Proofs for Propositional Linear-Time Logic

Mariusz Urbanski[1] Alexander Bolotov[2] Vasilyi Shangin[3] Oleg Grigoriev[3]

[1] Adam Mickiewicz University, Poznań, Poland
Mariusz.Urbanski@amu.edu.pl
[2] University of Westminster, UK
A.Bolotov@wmin.ac.uk
[3] Moscow State University, Russia
shangin@philos.msu.ru, grigoriev@philos.msu.ru

Abstract:

This paper presents a calculus of Socratic proofs for Propositional Linear-Time Logic (PLTL) and discusses potential automation of its proof search.

1 Introduction

Propositional Linear-Time Logic (PLTL) [6] gained various deductive constructions: axiomatic [5], tableau [11], resolution [4], and natural deduction [1]. In this paper we present a calculus of Socratic proofs for Propositional Linear-Time Logic (PLTL) [6], [3], [7] abbreviated as $\mathsf{PLTL_{SP}}$. The calculus is based upon the hypersequent calculus [2] and it fits into the framework of Socratic proofs by Wisniewski (cf. [8], [10] and [9]).

2 Logic PLTL$_T$

We utilise the language of PLTL which extends the language of Classical Propositional Calculus (CPC) by temporal operators: $\mathcal{U}$ (*until*) $\bigcirc$ (*at the next moment in time*), $\Box$ (*always in the future*), and $\Diamond$ (*at sometime in the future* or *eventually*). The semantics for the temporal part of the logic PLTL$_T$ is defined in the standard way over linear sequence of states, finite in the past, infinite in the future.

In order to formulate PLTL$_T$ we need to extend the language of PLTL with the following signs: $\vdash$, ?, 1 and 2. Intuitively, $\vdash$ stands for derivability relation and ? is a question-forming operator. The numerals 1 and 2 will be used to encode tree-structure of a Socratic transformation.

There are two disjoint categories of wffs: *declarative* wffs (d-wffs) and *erotetic* wffs (e-wffs), or questions. There are also two types of d-wffs: *atomic* d-wffs and *indexed d-wffs*. Atomic d-wffs are expressions of the form $S \vdash A$, where S is a finite sequence (possibly with repetitions) of PLTL-wffs, and A is a PLTL-wff, and if A is an empty formula, then S is a non-empty sequence. Indexed d-wffs are expressions of the form $S \vdash^n A$ or of the form $T \vdash^n$, where $S \vdash A$ and $T \vdash$ are atomic d-wffs of and n is a sequence of 1's or 2's, starting with 1. E-wffs, or questions are expressions of the form $?(\Phi)$, where Φ is a non-empty finite sequence of indexed atomic d-wffs (*constituents* of Φ).

In the formulation of rules we shall use the following classification of PLTL formulae to α and β types:

α	α_1	α_2
$A \wedge B$	A	B
$\neg(A \vee B)$	$\neg A$	$\neg B$
$\neg(A \to B)$	A	$\neg B$
$\Box A$	A	$\bigcirc\Box A$
$\neg\Diamond A$	$\neg A$	$\bigcirc\Box\neg A$
$\neg(A\mathcal{U}B)$	$\neg B$	$\neg(A \wedge \bigcirc(A\mathcal{U}B))$

β	β_1	β_2	β_1^*
$\neg(A \wedge B)$	$\neg A$	$\neg B$	A
$A \vee B$	A	B	$\neg A$
$A \to B$	$\neg A$	B	A
$\neg\Box A$	$\neg A$	$\bigcirc\Diamond\neg A$	A
$\Diamond A$	A	$\bigcirc\Diamond A$	$\neg A$
$A\mathcal{U}B$	B	$A \wedge \bigcirc(A\mathcal{U}B)$	$\neg B$

Rules for PLTL$_T$:

$$\mathbf{L}_\alpha : \quad \frac{?\,(\Phi; S\,'\,\alpha\,'\,T \vdash^n C; \Psi)}{?\,(\Phi; S\,'\,\alpha_1\,'\,\alpha_2\,'\,T \vdash^n C; \Psi)}$$

$$\mathbf{L}_\beta : \quad \frac{?\,(\Phi; S\,'\,\beta\,'\,T \vdash C; \Psi)}{?\,(\Phi; S\,'\,\beta_1\,'\,T \vdash^{n1} C; S\,'\,\beta_2\,'\,T \vdash^{n2} C; \Psi)}$$

$$\mathbf{L}_{\neg\neg} : \quad \frac{?\,(\Phi; S\,'\,\neg\neg A\,'\,T \vdash^n C; \Psi)}{?\,(\Phi; S\,'\,A\,'\,T \vdash^n C; \Psi)}$$

$$\mathbf{L}_{\neg\bigcirc} : \quad \frac{?\,(\Phi; S\,'\,\neg\bigcirc A\,'\,T \vdash^n C; \Psi)}{?\,(\Phi; S\,'\,\bigcirc\neg A\,'\,T \vdash^n C; \Psi)}$$

$$\mathbf{R}_\alpha : \quad \frac{?\,(\Phi; S \vdash^n \alpha; \Psi)}{?\,(\Phi; S \vdash^{n1} \alpha_1; S \vdash^{n2} \alpha_2; \Psi)}$$

$$\mathbf{R}_\beta : \quad \frac{?\,(\Phi; S \vdash^n \beta; \Psi)}{?\,(\Phi; S\,'\,\beta_1^* \vdash^n \beta_2; \Psi)}$$

$$\mathbf{R}_{\neg\neg} : \quad \frac{?\,(\Phi; S \vdash^n \neg\neg A; \Psi)}{?\,(\Phi; S \vdash^n A; \Psi)}$$

$$\mathbf{R}_{\neg\bigcirc} : \quad \frac{?\,(\Phi; S \vdash^n \neg\bigcirc A; \Psi)}{?\,(\Phi; S \vdash^n \bigcirc\neg A; \Psi)}$$

If none of the above rules is applicable to a PLTL formula B, such a formula is called *marked*. If all PLTL-formulas within an indexed formula $S \vdash^n A$ are marked, such a formula is called *a state*.

The following is a state-prestate rule:

$$\mathbf{S-P} : \quad \frac{?\,(\Phi; S \vdash^n A; \Psi)}{?\,(\Phi; S^\circ \vdash^n A^\circ; \Psi)}$$

where $S \vdash^n A$ is a state and S° (resp. A°) results from S (resp. A) by replacing all the formulas of the form $\bigcirc B$ with B and deleting all the remaining formulas. Every formula of the form $S^* \vdash^m A^*$, where n is an initial subsequence of m or m is an initial subsequence of n, is called *a pre-state* (cf. [11]).

Definition 1. *Let* $\mathbf{q} = \langle Q_1, \ldots, Q_r \rangle$ *be a finite sequence of questions of* $\mathbf{P}^*$. *Let* Q_g, Q_{h-1}, Q_h $(1 \leq g < h-1 \leq r)$ *be elements of the sequence* $\mathbf{q}$. *Let* $S_j \vdash^n A_j$ *be a constituent of* Q_g *and let* $S_k \vdash^m A_k$ *be a constituent of* Q_h *such that* $S_j = S_k$, $A_j = A_k$ *and the sequence* n *is an initial subsequence of the sequence* m. *Let* $S_l \vdash^i A_l$ *be a constituent of* Q_{h-1} *such that* $S_k \vdash^m A_k$ *is obtained from* $S_l \vdash^i A_l$ *by application of a PT*-rule. Then* $S_j \vdash^n A_j, \ldots, S_l \vdash^i A_l$ *form a* loop *(a sequence of atomic d-wffs of* $\mathbf{P}^*$ *... etc.), and* $S_k \vdash^m A_k$ *is called a* loop-generating formula.

Socratic transformations are sequences of questions that aim at deciding derivability of formuls from sets of formuls.

Definition 2. *A finite sequence* $\langle Q_1, \ldots, Q_r \rangle$ *of questions of* $\mathbf{P}^*$ *is a* Socratic transformation *of* $S \vdash A$ *iff the following conditions hold: (i)*$Q_1 = ?(S \vdash^1 A)$; *(ii)*Q_i *(where* $i = 2, \ldots, r$*) results from* Q_{i-1} *by applying a PT*-rule.*

Definition 3. *A constituent* ϕ *of a question* Q_i *is called* successful *iff one of the following holds: (a)* ϕ *is of the form* $T'B'U \vdash^n B$, *or (b)* ϕ *is of the form* $T'B'U'\neg B'W \vdash^n C$, *or (c)* ϕ *is of the form* $T'\neg B'U'B'W \vdash^n C$.

Definition 4. *A Socratic transformation* $\langle Q_1, \ldots, Q_r \rangle$ *of* $S \vdash A$ *is* completed *iff the for each constituent* ϕ *of* Q_r *at least one of the following conditions hold: (a) no rule is applicable to PLTL-formulas in* ϕ, *or (b)* ϕ *is successful, or (c)* ϕ *is a loop-generating formula.*

Definition 5. *A formula* B *is called an* eventuality *in* $S \vdash^n A$ *iff one of the following holds: (i)* B *is a term of* S *and there exists a PLTL-formula* C *such that* $B = \Diamond C$, *or (ii) there exists a PLTL-formula* C *such that* $B = A = \Box C$.

Definition 6. *A completed Socratic transformation* $\mathbf{q} = \langle Q_1, \ldots, Q_r \rangle$ *is a Socratic proof of* $S \vdash A$ *iff: (a) all the constituents of* Q_n *are successful, or (b) for each non-successful constituent* ϕ *of* Q_n, ϕ *is a loop-generating formula and the loop generated by* ϕ *contains a pre-state with an unfulfilled eventuality.*

The presented system is sound and complete. Proofs of these theorems involve construction of a canonical model with maximal consistent sets of formulae as its states.

3 Examples

In the examples below by highlighting we indicate a formula which is analyzed at the current step. Double underlining of a formula reflects that it is a state. The question following the one containing a state is obtained by state-prestate rule.

Example 1

1. $?(\vdash^1 \Box p \to p)$
2. $?(\Box p \vdash^1 p)$
3. $?(p, \bigcirc\Box p \vdash^1 p)$

Example 2

1. $?(\vdash^1 \Box p \to \bigcirc p)$
2. $?(\Box p \vdash^1 \bigcirc p)$
3. $?(\underline{\underline{p, \bigcirc\Box p \vdash^1 \bigcirc p}})$
4. $?(\Box p \vdash^1 p)$
5. $?(p, \bigcirc\Box p \vdash^1 p)$

References

[1] O. Grigoryev A. Basukoski, A. Bolotov and V. Shangin. Natural deduction system for linear time temporal logic . *Logical investigations*, (13):71–95, 2004.

[2] A. Avron. *The Method of Hypersequents in the Proof Theory of Propositional Non-Classical Logics*, pages 1–32. Logic: Foundations to Applications. Clarendon Press, 1996.

[3] M. Finger, D. M. Gabbay, and M. Reynolds. *Advanced Tense Logic*, volume 7 of *Handbook of Philosophical Logic*, pages 43–203. Springer, 2002.

[4] Michael Fisher, Clare Dixon, and Martin Peim. Clausal temporal resolution. *ACM Trans. Comput. Log.*, 2(1):12–56, 2001.

[5] D. M. Gabbay, A. Pnueli, S. Shelah, and J. Stavi. On the temporal analysis of fairness. In *7th ACM Symposium on Principles of Programming Languages*, pages 163–173, 1980.

[6] A. Pnueli. The temporal logic of programs. In *Proceedings of the 18th Symposium on Foundations of Computer Science*, pages 46–57, 1977.

[7] A. P. Sistla and E. M. Clarke. The complexity of Propositional Linear Temporal Logic. *Journal of the Association for Computing Machinery*, 32(3):733–749, 1985.

[8] A. Wiśniewski. Socratic Proofs. *Journal of Philosophical Logic*, 33(2):299–326, 2004.

[9] A. Wiśniewski and V. Shangin. Socratic proofs for quantifiers. *Journal of Philosophical Logic*, 35(2):147–178, 2006.

[10] A. Wiśniewski, G. Vanackere, and D. Leszczyńska. Socratic Proofs and Paraconsistency. A Case Study. *Studia Logica*, 80:431–466, 2005.

[11] P. Wolper. The tableau method for temporal logic: An overview. *Logique et Analyse*, 28:119–136, 1985.