Analysis and Evaluation of the Family of Sign Adaptive Algorithms

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Analysis and Evaluation of the Family of Sign Adaptive Algorithms

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Abstract

In this thesis, four novel sign adaptive algorithms proposed by the author were analyzed and evaluated for floating-point arithmetic operations. These four algorithms include Sign Regressor Least Mean Fourth (SRLMF), Sign Regressor Least Mean Mixed-Norm (SRLMMN), Normalized Sign Regressor Least Mean Fourth (NSRLMF), and Normalized Sign Regressor Least Mean Mixed-Norm (NSRLMMN). The performance of the latter three algorithms has been analyzed and evaluated for real-valued data only. While the performance of the SRLMF algorithm has been analyzed and evaluated for both cases of real- and complex-valued data.

Additionally, four sign adaptive algorithms proposed by other researchers were also analyzed and evaluated for floating-point arithmetic operations. These four algorithms include Sign Regressor Least Mean Square (SRLMS), Sign-Sign Least Mean Square (SSLMS), Normalized Sign-Error Least Mean Square (NSLMS), and Normalized Sign Regressor Least Mean Square (NSRLMS). The performance of the latter three algorithms has been analyzed and evaluated for both cases of real- and complex-valued data. While the performance of the SRLMS algorithm has been analyzed and evaluated for complex-valued data only.

The framework employed in this thesis relies on energy conservation approach. The energy conservation framework has been applied uniformly for the evaluation of the performance of the aforementioned eight sign adaptive algorithms proposed by the author and other researchers. In other words, the energy conservation framework stands out as a common theme that runs throughout the treatment of the performance of the aforementioned eight algorithms.

Some of the results from the performance evaluation of the four novel sign adaptive algorithms proposed by the author, namely SRLMF, SRLMMN, NSRLMF, and NSRLMMN are as follows. It was shown that the convergence performance of the SRLMF and SRLMMN algorithms for real-valued data was similar to those of the Least Mean Fourth (LMF) and Least Mean Mixed-Norm (LMMN) algorithms, respectively. Moreover, it was also shown that the NSRLMF and NSRLMMN algorithms exhibit a compromised convergence performance for real-valued data as compared to the Normalized Least Mean Fourth (NLMF) and Normalized Least Mean Mixed-Norm (NLMMN) algorithms, respectively.

Some misconceptions among biomedical signal processing researchers concerning the implementation of adaptive noise cancelers using the Sign-Error Least Mean Fourth (SLMF), Sign-Sign Least Mean Fourth (SSLMF), and their variant algorithms were also removed.

Finally, three of the novel sign adaptive algorithms proposed by the author, namely SRLMF, SRLMMN, and NSRLMF have been successfully employed by other researchers and the author in applications ranging from power quality improvement in the distribution system and multiple artifacts removal from various physiological signals such as ElectroCardioGram (ECG) and ElectroEncephaloGram (EEG).

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Author's Original Contributions/Publications

The list of publications submitted for the Doctor of Philosophy by Published Work is presented below and will be referred to as [P1], [P2], etc. throughout this thesis except in the list of references and their respective citations:

[P1] M. M. U. Faiz, A. Zerguine, and A. Zidouri, "Analysis of the sign regressor least mean fourth adaptive algorithm," *EURASIP Jour. on Advances in Signal Processing,* vol. 2011, art. no. 373205, pp. 1–12, Jan. 2011, DOI: https://doi.org/10.1155/2011/373205

A novel adaptive algorithm called the Sign Regressor Least Mean Fourth (SRLMF) algorithm proposed by the author that reduces the computational cost and complexity while maintaining good performance is analyzed and evaluated. New expressions are derived for the steady-state Mean Square Error (MSE) of the SRLMF algorithm in a stationary environment. A sufficient condition for the convergence in the mean of the SRLMF algorithm is derived. In addition, new expressions are obtained for the tracking MSE of the SRLMF algorithm in a nonstationary environment and consequently an optimum value of the step-size of the SRLMF algorithm is obtained. A comparison between the convergence performance of the SRLMF algorithm and the classical Least Mean Fourth (LMF) algorithm is also presented. Finally, simulations are carried out to corroborate the new theoretical findings of the SRLMF algorithm.

[P2] M. M. U. Faiz and A. Zerguine, "Analysis of the complex sign regressor least mean fourth adaptive algorithm," *in Proc. of the 2nd IEEE Int. Conf. on Signal and Image Processing Applications (ICSIPA 2011),* Kuala Lumpur, Malaysia, pp. 553–555, Nov. 2011, DOI: https://doi.org/10.1109/ICSIPA.2011.6144115

New expressions are derived for the steady-state and tracking MSE of the novel complex SRLMF algorithm proposed by the author. In addition, a novel expression for the optimum step-size of the complex SRLMF algorithm is also obtained. Moreover, a comparison between the convergence performance of the SRLMF algorithm and the classical LMF algorithm for complex-valued data is also presented. Finally, simulations are carried out to substantiate the new theoretical findings of the complex SRLMF algorithm.

[P3] M. M. U. Faiz and A. Zerguine, "Adaptive channel equalization using the sign regressor least mean fourth algorithm," *in Proc. of the 1st IEEE Saudi Int. Electronics, Communications and Photonics Conf. (SIECPC 2011),* Riyadh, Saudi Arabia, pp. 1–4, Apr. 2011, DOI: https://doi.org/10.1109/SIECPC.2011.5876986

In this paper, the performance of the novel SRLMF algorithm proposed by the author in [P1] and the classical LMF algorithm is investigated in an adaptive channel equalization scenario. The simulation results indicate that both the SRLMF and LMF algorithms exhibit similar Bit Error Rate (BER) performance. Moreover, the results show that the SRLMF algorithm has a slight performance degradation in terms of convergence behavior when compared with the LMF algorithm in this setting.

[P4] M. M. U. Faiz and A. Zerguine, "On the convergence, steady-state, and tracking analysis of the SRLMMN algorithm," ***in Proc. of the 23rd European Signal Processing Conf. (EUSIPCO 2015),*** **Nice, France, pp. 2691–2695, Aug.-Sep. 2015, DOI: https://doi.org/10.1109/EUSIPCO.2015.7362873**

A novel adaptive algorithm called the Sign Regressor Least Mean Mixed-Norm (SRLMMN) algorithm is proposed by the author as an alternative to the classical Least Mean Mixed-Norm (LMMN) algorithm. New analytical expressions are derived to describe the convergence, steady-state, and tracking behaviors of the SRLMMN algorithm. A comparison between the convergence performance of the SRLMMN and LMMN algorithms is also presented. Moreover, the behavior of the SRLMMN algorithm is also compared for different values of the mixing parameter and step-size. Finally, extensive simulations are carried out to substantiate the new theoretical findings of the SRLMMN algorithm.

[P5] M. M. U. Faiz and I. Kale, "Removal of multiple artifacts from ECG signal using cascaded multistage adaptive noise cancellers," ***Array,*** **vol. 14, art. no. 100133, pp. 1–9, July 2022, DOI: https://doi.org/10.1016/j.array.2022.100133**

A novel cascaded 4-stage adaptive noise canceller is proposed for the removal of four artifacts present in the ElectroCardioGram (ECG) signal, namely baseline wander, motion artifacts, muscle artifacts, and 60 Hz Power Line Interference (PLI). One unique and powerful feature of the proposed novel cascaded 4-stage adaptive noise canceller is that it employs only those adaptive algorithms in the four stages, which are shown to be effective in removing the respective ECG artifacts as mentioned above. Such a scheme has not been investigated before in the open literature. The proposed novel cascaded 4-stage adaptive noise canceller employing the shortlisted LMMN, LMF, LMMN, LMF algorithms outperforms those that employ the same algorithm in the four stages.

[P6] M. M. U. Faiz and A. Zerguine, "The ε-Normalized Sign Regressor Least Mean Fourth (NSRLMF) adaptive algorithm," ***in Proc. of the 11th IEEE Int. Conf. on Information Sciences, Signal Processing and their Applications (ISSPA 2012),*** **Montreal, QC, Canada, pp. 339–342, July 2012, DOI: https://doi.org/10.1109/ISSPA.2012.6310571**

A novel adaptive algorithm called the Normalized Sign Regressor Least Mean Fourth (NSRLMF) algorithm proposed by the author is analyzed and evaluated. New expressions are derived for the steady-state MSE of the NSRLMF algorithm. Moreover, a comparison between the convergence performance of the NSRLMF algorithm and the Normalized Least Mean Fourth (NLMF) algorithm is also presented. Finally, simulations are carried out to corroborate the new theoretical findings of the NSRLMF algorithm.

[P7] M. M. U. Faiz and A. Zerguine, "Convergence and tracking analysis of the ε-NSRLMF algorithm," ***in Proc. of the 38th IEEE Int. Conf. on Acoustics, Speech, and Signal Processing (ICASSP 2013),*** **Vancouver, BC, Canada, pp. 5657–5660, May 2013, DOI: https://doi.org/10.1109/ICASSP.2013.6638747**

The convergence and tracking behaviors of the novel NSRLMF algorithm proposed by the author in [P6] are analyzed and evaluated in the presence of both white Gaussian and correlated Gaussian regressors. Furthermore, the stability bound on the step-size of the NSRLMF algorithm to ensure convergence in the mean, which also leads us to the mean convergence of the Normalized Sign Regressor Least Mean Mixed-Norm (NSRLMMN) algorithm is derived. Finally, simulation results are conducted to corroborate the new theoretical findings of the NSRLMF algorithm.

[P8] M. M. U. Faiz and A. Zerguine, "Convergence analysis of the ε NSRLMMN algorithm," ***in Proc. of the 20th European Signal Processing Conf. (EUSIPCO 2012),*** **Bucharest, Romania, pp. 235–239, Aug. 2012, ISBN: 978-1-4673-1068-0**

A novel adaptive algorithm called the NSRLMMN algorithm proposed by the author is analyzed and evaluated. New expressions are derived for the steady-state MSE of the NSRLMMN algorithm. A sufficient condition for the convergence in the mean of the NSRLMMN algorithm is derived. A comparison between the convergence performance of the NSRLMMN algorithm and the Normalized Least Mean Mixed-Norm (NLMMN) algorithm is also presented. Moreover, the behavior of the NSRLMMN algorithm is also compared for different values of the mixing parameter and step-size. Finally, simulations are carried out to corroborate the new theoretical findings of the NSRLMMN algorithm.

[P9] M. M. U. Faiz and A. Zerguine, "Tracking analysis of the ε-NSRLMMN algorithm," ***in the Conf. Record of the 46th Asilomar Conf. on Signals, Systems, and Computers (Asilomar 2012),*** **Pacific Grove, CA, USA, pp. 816–819, Nov. 2012, DOI: https://doi.org/10.1109/ACSSC.2012.6489127**

New expressions are derived for the optimum step-size and the corresponding minimum value of the tracking MSE of the novel NSRLMMN algorithm proposed by the author in [P8]. Moreover, a comparison between the convergence performance of the NSRLMMN and LMMN algorithms indicates faster convergence of the NSRLMMN algorithm. Finally, extensive simulations are carried out to substantiate the new theoretical findings of the NSRLMMN algorithm in the presence of both white Gaussian and correlated Gaussian regressors.

[P10] M. M. U. Faiz and A. Zerguine, "On the steady-state and tracking analysis of the complex SRLMS algorithm," ***in Proc. of the 22nd European Signal Processing Conf. (EUSIPCO 2014),*** **Lisbon, Portugal, pp. 751–754, Sep. 2014, E-ISBN: 978-0-9928-6261-9**

New expressions are derived for the steady-state MSE, the optimum step-size, and the corresponding minimum value of the tracking MSE of the Sign Regressor Least Mean Square (SRLMS) algorithm for the case of complex-valued data. Moreover, a comparison between the convergence performance of the SRLMS algorithm and the classical Least Mean Square (LMS) algorithm for complex-valued data is also presented. Finally, simulation results are presented to support the new analytical findings of the complex SRLMS algorithm in the presence of both white Gaussian and correlated Gaussian regressors.

[P11] M. M. U. Faiz and A. Zerguine, "Steady-State and tracking analysis of the SSLMS algorithm," ***in Proc. of the 15th IEEE Int. Multi-Conf. on Systems, Signals & Devices (SSD 2018),*** **Hammamet, Tunisia, pp. 45–48, Mar. 2018, DOI: https://doi.org/10.1109/SSD.2018.8570395**

New expressions are derived for the steady-state MSE, the optimum step-size, and the corresponding minimum value of the tracking MSE of the Sign-Sign Least Mean Square (SSLMS) algorithm for the case of real-valued data. Moreover, a comparison between the convergence performance of the SSLMS algorithm and the classical LMS algorithm for real-valued data is also presented. Finally, simulation results are presented to support the new analytical findings of the SSLMS algorithm in the presence of both white Gaussian and correlated Gaussian regressors.

[P12] M. M. U. Faiz and A. Zerguine, "Analysis of the SSLMS algorithm for complex-valued data," ***in Proc. of the 16th IEEE Int. Multi-Conf. on Systems, Signals & Devices (SSD 2019),*** **Istanbul, Turkey, pp. 262–265, Mar. 2019, DOI: https://doi.org/10.1109/SSD.2019.8893215**

New expressions are derived for the steady-state MSE, the optimum step-size, and the corresponding minimum value of the tracking MSE of the SSLMS algorithm for the case of complex-valued data. Moreover, a comparison between the convergence performance of the SSLMS algorithm and the classical LMS algorithm for complex-valued data is also presented. Finally, simulation results are presented to support the new analytical findings of the complex SSLMS algorithm in the presence of both white Gaussian and correlated Gaussian regressors.

[P13] M. M. U. Faiz and A. Zerguine, "Insights into the convergence and steady-state behaviors of the SLMF and its variants," ***in Proc. of the 12th IEEE Int. Multi-Conf. on Systems, Signals & Devices (SSD 2015),*** **Mahdia, Tunisia, pp. 1–4, Mar. 2015, DOI: https://doi.org/10.1109/SSD.2015.7348094**

The convergence and steady-state behaviors of the Sign-Error Least Mean Fourth (SLMF), Sign-Sign Least Mean Fourth (SSLMF), Normalized Sign-Error Least Mean Fourth (NSLMF), and Normalized Sign-Sign Least Mean Fourth (NSSLMF) algorithms are investigated for both cases of real- and complex-valued data. Moreover, the equivalence algorithms of the Block-Based Normalized Sign-Error Least Mean Fourth (BBNSLMF) and Block-Based Normalized Sign-Sign Least Mean Fourth (BBNSSLMF) algorithms are also reported.

[P14] M. M. U. Faiz, "Comments on "Efficient signal conditioning techniques for brain activity in remote health monitoring network"," ***IEEE Sensors Jour.,*** **vol. 15, no. 9, pp. 5349–5350, Sep. 2015, DOI: https://doi.org/10.1109/JSEN.2015.2431260**

The main purpose of this paper is to remove misconceptions among biomedical signal processing researchers concerning the implementation of adaptive noise cancelers using the SLMF, SSLMF, and their variant algorithms. It was shown that the filter weights update equations of the SLMF and SSLMF algorithms for real-valued data are exactly identical to those of the Sign-Error Least Mean Square (SLMS) and SSLMS algorithms, respectively.

[P15] M. M. U. Faiz and A. Zerguine, "The ε-Normalized Sign Regressor Least Mean Square (NSRLMS) adaptive algorithm," ***in Proc. of the 2nd IEEE Int. Conf. on Signal and Image Processing Applications (ICSIPA 2011),*** **Kuala Lumpur, Malaysia, pp. 556–558, Nov. 2011, DOI: https://doi.org/10.1109/ICSIPA.2011.6144114**

New expressions are derived for the steady-state and tracking MSE of the Normalized Sign Regressor Least Mean Square (NSRLMS) algorithm for the case of complex-valued data. Finally, it is shown that simulations performed for both cases of white Gaussian and correlated Gaussian regressors substantiate very well the new theoretical findings of the NSRLMS algorithm.

[P16] M. M. U. Faiz and A. Zerguine, "A note on NSRLMS, NSRLMF, and NSRLMMN adaptive algorithms," ***in Proc. of the 15th IEEE Int. Multi-Conf. on Systems, Signals & Devices (SSD 2018),*** **Hammamet, Tunisia, pp. 40–44, Mar. 2018, DOI: https://doi.org/10.1109/SSD.2018.8570653**

Expressions for the steady-state MSE, the optimum step-size, and the corresponding minimum tracking MSE of the NSRLMS, NSRLMF, and NSRLMMN algorithms for the case of real-valued data are compared. These expressions are available in the open literature for the latter two algorithms. Thus, in order to compare these three algorithms, new expressions are derived for the aforementioned three parameters of the NSRLMS algorithm. Finally, simulations are carried out to substantiate the new analytical results of the NSRLMS algorithm in the presence of both white Gaussian and correlated Gaussian regressors.

[P17] M. M. U. Faiz and A. Zerguine, "A steady-state analysis of the ε-Normalized Sign-Error Least Mean Square (NSLMS) adaptive algorithm," ***in the Conf. Record of the 45th Asilomar Conf. on Signals, Systems, and Computers (Asilomar 2011),*** **Pacific Grove, CA, USA, pp. 538–541, Nov. 2011, DOI: https://doi.org/10.1109/ACSSC.2011.6190059**

New expressions are derived for the steady-state MSE of the Normalized Sign-Error Least Mean Square (NSLMS) algorithm for both cases of real- and complex-valued data. It is interesting to note that the analytical expressions derived for the steady-state MSE of the NSLMS algorithm for real- and complex-valued data are identical except for a scaling factor. Finally, simulations are carried out to corroborate the new theoretical findings of the NSLMS algorithm in the presence of both white Gaussian and correlated Gaussian regressors.

[P18] M. M. U. Faiz, A. Zerguine, S. M. Asad, and K. Mahmood, "Tracking MSE performance analysis of the ε-NSLMS algorithm," ***in Proc. of the 2nd IEEE Int. Conf. on Communications, Signal Processing and their Applications (ICCSPA 2015),*** **Sharjah, UAE, pp. 1–4, Feb. 2015, DOI: https://doi.org/10.1109/ICCSPA.2015.7081323**

The tracking behavior of the NSLMS algorithm is analyzed and evaluated in the presence of both white Gaussian and correlated Gaussian regressors. Moreover, novel generic analytical expressions are derived for the optimal step-size and the corresponding minimum tracking MSE of the NSLMS algorithm for both the real- and complex-valued data cases. Additionally, a comparison between the convergence behaviors of the NSLMS algorithm and the Normalized Least Mean Square (NLMS) algorithm is also discussed. Finally, simulation results to corroborate the new theoretical findings of the NSLMS algorithm are presented.

Acknowledgments

I would like to express my profound gratitude and deep appreciation to Prof. Izzet Kale for agreeing to supervise my Ph.D. by published work thesis. It has been a great learning experience for me to work under his supervision. I have learnt so many skills under him such as biomedical signal processing, floating-to-fixed point conversion, fixed-point adaptive noise cancellers, etc. I look forward to continuing my learning experience with Prof. Izzet in the future too. I would also like to thank him for spending his precious time helping me present this thesis in a way that could be valuable to other researchers.

I would also like to express my profound gratitude and deep appreciation to Prof. Azzedine Zerguine for playing the role of a supervisor in all the publications listed in the author's original contributions except in [P5] and [P14]. I was very fortunate to have him as my supervisor over the many years of my research on adaptive filtering algorithms and their applications. It has been a great learning experience for me while carrying out my research work under his supervision.

I would also like to thank my Second Supervisor Dr. Saumya Kareem Reni for her kind advice and support during the submission of my Ph.D. by published work thesis.

Last but not least, I would also like to thank my parents, my younger brothers, my wife, my daughter, my father-in-law, my mother-in-law, my brothers-in-law, my sister-in-law, all my friends, and colleagues for their constant support and encouragement.

Author's Declaration

I declare that all the 18 publications listed in the author's original contributions is my own work. Furthermore, I conducted the research by myself, while Prof. Azzedine Zerguine played a role of a supervisor in all the publications listed in the author's original contributions except in [P5] and [P14].

In [P5], I conducted the research by myself, while Prof. Izzet Kale played the role of a supervisor. Finally, I conducted the research by myself in [P14] with no supervision as it is a single author publication.

Mohammed Mujahid Ulla Faiz
01/12/2022

List of Abbreviations

AWGN	Additive White Gaussian Noise
BBNSLMF	Block-Based Normalized Sign-Error Least Mean Fourth
BBNSRLMF	Block-Based Normalized Sign Regressor Least Mean Fourth
BBNSSLMF	Block-Based Normalized Sign-Sign Least Mean Fourth
BER	Bit Error Rate
BPSK	Binary Phase Shift Keying
ECG	ElectroCardioGram
EEG	ElectroEncephaloGram
GER	Globally Engaged Research
LMF	Least Mean Fourth
LMMN	Least Mean Mixed-Norm
LMS	Least Mean Square
LSRA	Leaky Sign Regressor Algorithm
LSRLMMN	Leaky Sign Regressor Least Mean Mixed-Norm
MSD	Mean Square Deviation
MSE	Mean Square Error
NLMF	Normalized Least Mean Fourth
NLMMN	Normalized Least Mean Mixed-Norm
NLMS	Normalized Least Mean Square
NSA	Normalized Sign Algorithm
NSLMF	Normalized Sign-Error Least Mean Fourth
NSLMS	Normalized Sign-Error Least Mean Square
NSRA	Normalized Sign Regressor Algorithm
NSRLMF	Normalized Sign Regressor Least Mean Fourth
NSRLMMN	Normalized Sign Regressor Least Mean Mixed-Norm
NSRLMS	Normalized Sign Regressor Least Mean Square
NSSLMF	Normalized Sign-Sign Least Mean Fourth
NSSLMS	Normalized Sign-Sign Least Mean Square
PLI	Power Line Interference
PSK	Phase Shift Keying
QAM	Quadrature Amplitude Modulation
QPSK	Quadrature Phase Shift Keying
RLS	Recursive Least-Squares
SA	Sign Algorithm
SLMF	Sign-Error Least Mean Fourth
SLMS	Sign-Error Least Mean Square
SNR	Signal to Noise Ratio
SRA	Sign Regressor Algorithm
SRLMF	Sign Regressor Least Mean Fourth
SRLMMN	Sign Regressor Least Mean Mixed-Norm
SRLMS	Sign Regressor Least Mean Square
SSA	Sign-Sign Algorithm
SSLMF	Sign-Sign Least Mean Fourth
SSLMS	Sign-Sign Least Mean Square

VSLMS	Variable Step-size Least Mean Square

1 Introduction

1.1 Adaptive Filters

When the filter is required to operate in a stationary environment, wherein the statistics of the signal to be processed are known, the use of Wiener filter provides a solution, which is optimum in the mean square error sense. However, when the filter is required to operate in a nonstationary environment, wherein the statistics of the signal to be processed are unknown, the use of an adaptive filter offers an attractive solution to the problem. In a nonstationary environment, adaptive filters provide significant improvement in performance over fixed filters [1].

An adaptive filter has the ability of adapting its characteristics in order to achieve the desired objectives. Adaptation is accomplished automatically by adjusting the filter coefficients or filter weights in accordance with the input data. Thus, making an adaptive filter in reality a nonlinear device as it does not obey the principle of superposition [1].

1.2 Adaptive Filtering Algorithms

An adaptive filter relies on a recursive algorithm for its operation. The algorithm starts from some predetermined set of initial conditions. In a stationary environment, the algorithm converges to the optimum Weiner solution after some successive iterations. In a nonstationary environment, the algorithm offers a tracking ability, wherein it can track time variations in the input data statistics [1], [2].

A number of adaptive algorithms have been reported in the open literature in order to adjust the filter coefficients. Some of the most well-known algorithms include Least Mean Square (LMS) [3]–[5], Least Mean Fourth (LMF) [6]–[9], Least Mean Mixed-Norm (LMMN) [10]–[12], Sign Regressor Least Mean Square (SRLMS) [13]–[16], Sign-Error Least Mean Square (SLMS) [17]–[25], Sign-Sign Least Mean Square (SSLMS) [26]–[31], Sign Regressor Least Mean Fourth (SRLMF) [32], [33], Sign Regressor Least Mean Mixed-Norm (SRLMMN) [34] etc. The latter two novel algorithms, namely SRLMF and SRLMMN were proposed, analyzed, and evaluated by the author.

1.3 Applications of Adaptive Filters

Adaptive filters have been successfully employed in many diverse fields such as biomedical engineering, communications, control systems, radar, seismic, and sonar signal processing, etc. The main difference among the various applications arises in the manner in which the desired response is extracted. On this basis, adaptive filters are classified into four basic classes, namely modelling, inverse modelling, interference/noise cancellation, and linear prediction [1], [2].

In [32]–[46], system identification, an application of modelling, has been implemented to evaluate the performance of the respective algorithms discussed in these publications. In [47], channel equalization, an application of inverse modelling, has been implemented to evaluate the performance of the respective algorithms discussed in this particular publication. In [48], both single-stage and cascaded interference/noise cancellation adaptive filter structures have been implemented to evaluate the performance of the respective algorithms discussed in this particular publication.

Therefore, only modelling, inverse modelling, and both single-stage and cascaded interference/noise cancellation adaptive filter structures are briefly discussed below. Moreover, all the simulations in the publications listed in the author's original contributions except in [49] are performed by the author using MathWorks' MATLAB software (see Appendix A for a sample MATLAB program). It should be noted that there were no simulation results reported in [49] as it was a comments article.

1.3.1 Modelling

The problem of modelling in the context of adaptive filters is depicted in Figure 1.1. The aim is to estimate the parameters of an unknown system or plant. Both the unknown system and the adaptive filter are driven by the same input $\mathbf{u}_i$. v_i is the additive noise. The adaptive filter output y_i is subtracted from the desired signal d_i. The resulting error signal e_i is used to update the adaptive filter coefficients such that the error signal gets minimized iteratively [1], [2].

Note that throughout this thesis, scalar quantities such as v_i are denoted by lowercase letters, vector quantities such as $\mathbf{u}_i$ are denoted by boldfaced lowercase letters, and matrices such as regressor covariance matrix $\mathbf{R}$ are denoted by boldfaced uppercase letters.

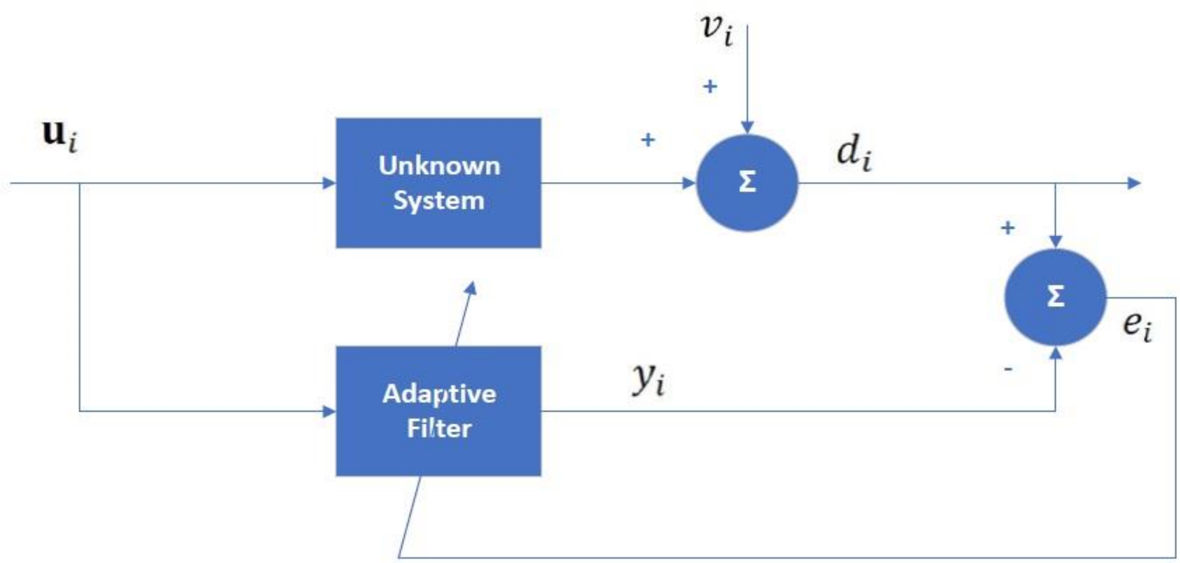

Figure 1.1: Modelling scenario.

1.3.2 Inverse Modelling

The problem of inverse modelling in the context of adaptive filters is depicted in Figure 1.2. The most widely used application of inverse modelling, also known as deconvolution, is in communications wherein an inverse model, also called an equalizer [50], is employed to mitigate the effect of channel distortion. At convergence, the adaptive filter has a best transfer function equal to the reciprocal of the unknown system's or plant's transfer function, such that the combination of the two constitutes an ideal transmission medium. In Figure 1.2, a delayed version of the system input s_i forms the desired signal d_i for the adaptive filter [1], [2].

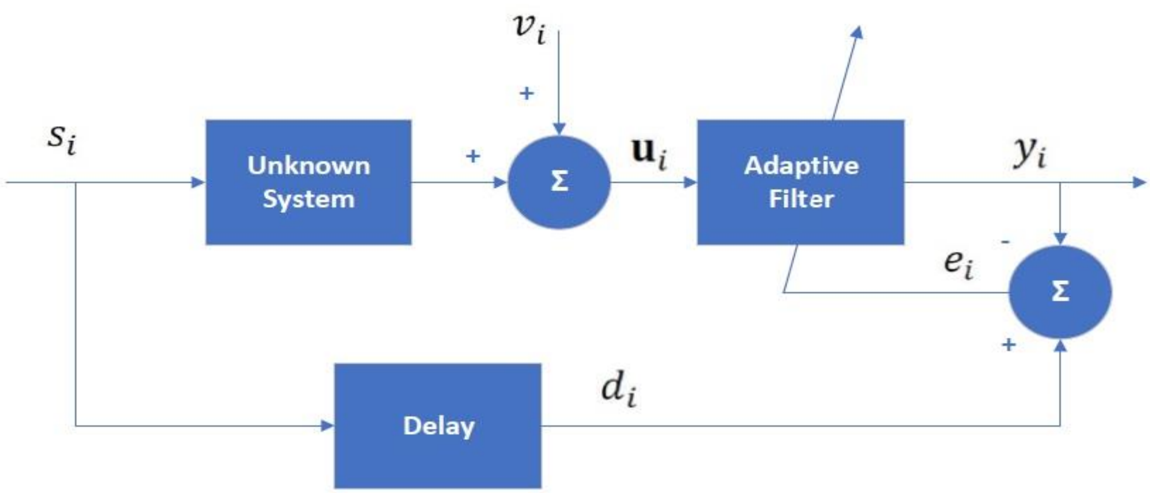

Figure 1.2: Inverse modelling scenario.

1.3.3 Interference/Noise Cancellation

The problem of interference/noise cancellation in the context of adaptive filters is depicted in Figure 1.3. In this application, the adaptive filter structure shown below is used to cancel the interference/noise present in the corrupted primary input d_i, which contains the desired signal plus interference/noise [51]. The secondary or reference input $\mathbf{u}_i$ contains the reference interference/noise that is correlated only with the interference/noise present in the corrupted primary input d_i, the adaptive filter is adjusted so that an estimate of the interference/noise that is present in the corrupted primary input d_i appears at its output y_i, and e_i is the filtered signal free from interference/noise [1], [2].

1.3.4 Cascaded 2-Stage Adaptive Noise Cancellation

A cascaded 2-stage adaptive noise canceller is shown in Figure 1.4. As can be seen from this figure d_{i1} forms the corrupted primary input of the first adaptive noise canceller, d_{i1} contains the desired signal plus the two noise signals n_1 and n_2, $\mathbf{u}_{i1}$ forms the reference input of the first adaptive noise canceller, $\mathbf{u}_{i1}$ contains the first reference noise signal that is correlated only with the first noise signal n_1 present in the corrupted primary input d_{i1}, $\mathbf{u}_{i2}$ forms the reference input of the second adaptive noise canceller, $\mathbf{u}_{i2}$ contains the second reference noise signal that is correlated only with the second noise signal n_2 present in the corrupted primary input d_{i1}, y_{i1} and y_{i2} are the respective adaptive filter outputs, e_{i1} is the partially corrupted signal free from the first noise signal n_1, e_{i1} will act as the partially corrupted primary input d_{i2} to the second adaptive noise canceller, and e_{i2} is the filtered signal free from both noise signals n_1 and n_2.

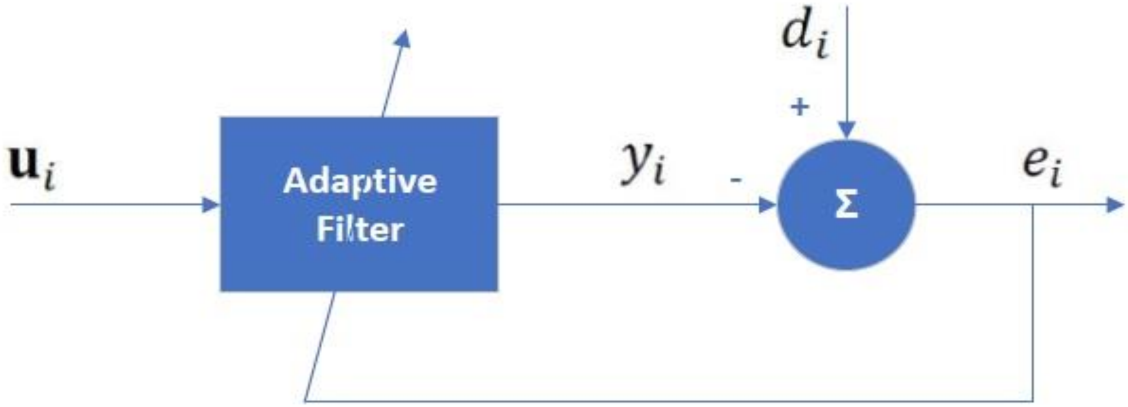

Figure 1.3: Interference/Noise cancellation scenario.

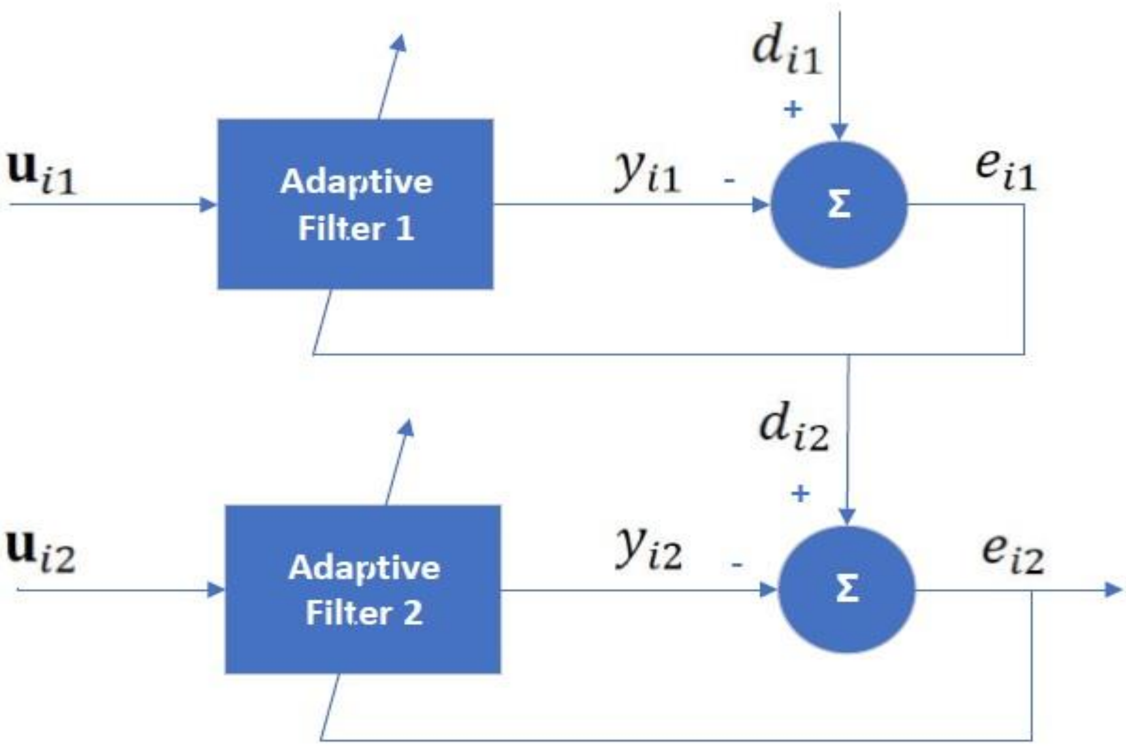

Figure 1.4: Cascaded 2-stage adaptive noise cancellation scenario.

1.4 Advantages of Sign Adaptive Algorithms

Reduction in the complexity of the LMS algorithm has always received attention in the area of adaptive filtering. This reduction is usually done by clipping either the input data or the estimation error or both in order to reduce the number of multiplications necessary at each algorithm iteration. The clipping of the input data or the estimation error or both is accomplished by the application of signum function.

The algorithm based on clipping of the input data is known as the Sign Regressor Algorithm (SRA), the algorithm based on clipping of the estimation error is known as the sign-error algorithm or more commonly the Sign Algorithm (SA), and the algorithm based on clipping of both the input data and the estimation error is known as the Sign-Sign Algorithm (SSA).

The aforementioned three sign adaptive algorithms result in a performance loss when compared with the LMS algorithm. However, significant reduction in computational cost and simplified hardware implementation can justify this performance loss in applications requiring reduced implementation costs.

1.5 Real and Complex Forms of Sign Adaptive Algorithms

There are certain applications wherein the adaptive filter input and its desired signal are in complex-valued form. For example, in digital data transmission, where the most widely employed signalling techniques are Phase Shift Keying (PSK) and Quadrature Amplitude Modulation (QAM). In this particular application, the baseband signal comprises of two separate components which are the real and imaginary parts of a complex-valued signal. Thus, we find cases where the formulation of the adaptive filtering algorithms must be given in terms of complex-valued variables. It should be noted that real adaptive filters are special cases of complex adaptive filters [1], [2].

Some of the sign adaptive algorithms such as the SRLMF algorithm has been analyzed and evaluated by the author for both cases of real- and complex-valued data in [32], [33].

1.6 Objectives

The first objective of this thesis was to propose, analyze, and evaluate various novel sign adaptive algorithms that exhibit good convergence rate with respect to their respective counterparts while maintaining all the advantages of the sign adaptive algorithms. This objective was achieved by analyzing and evaluating four novel sign adaptive algorithms proposed by the author, namely SRLMF [32], [33], SRLMMN [34], Normalized Sign Regressor Least Mean Fourth (NSRLMF) [35], [36], and Normalized Sign Regressor Least Mean Mixed-Norm (NSRLMMN) [37], [38].

In [32], it was shown that the convergence rate of the SRLMF algorithm was similar to that of the LMF algorithm. In [34], it was shown that the convergence rate of the SRLMMN algorithm was similar to that of the LMMN algorithm. In [35], it was shown that the NSRLMF algorithm has an improved convergence rate as compared to the LMF and SRLMF algorithms as per the expectations. However, the NSRLMF algorithm exhibits a compromised convergence rate as compared to the Normalized Least Mean Fourth (NLMF) algorithm as shown in [35] and [36]. In [37] and [38], it was shown that the NSRLMMN algorithm has an improved convergence rate as compared to the LMMN and SRLMMN algorithms as per the expectations. However, the NSRLMMN algorithm exhibits a compromised convergence rate as compared to the Normalized Least Mean Mixed-Norm (NLMMN) algorithm as shown in [37].

The second objective was to analyze and evaluate the performance of various sign adaptive algorithms for complex-valued data. This objective was achieved by analyzing and evaluating the performance of the SRLMS [39], SSLMS [41], SRLMF [33], Normalized Sign-Error Least Mean Square (NSLMS) [44], [45], and Normalized Sign Regressor Least Mean Square (NSRLMS) [42] algorithms.

The third objective was to analyze and evaluate the performance of various sign adaptive algorithms for real-valued data. This objective was achieved by analyzing and evaluating the performance of the SSLMS [40], SRLMF [32], SRLMMN [34], NSLMS [44], [45], NSRLMS [43], NSRLMF [35], [36], and NSRLMMN [37], [38] algorithms.

Note that the SRLMS, SSLMS, NSLMS, and NSRLMS algorithms mentioned in the second and third objectives were proposed and analyzed using different methods by other researchers.

Finally, the fourth objective was to remove misconceptions among biomedical signal processing researchers concerning the implementation of adaptive noise cancelers using the Sign-Error Least Mean Fourth (SLMF), Sign-Sign Least Mean Fourth (SSLMF), and their variant algorithms. This objective was achieved in [46], [49].

1.7 Methodology

The framework employed in the publications listed in the author’s original contributions in [32]–[45], relies on energy conservation approach as described in [52]. The energy conservation framework has been applied uniformly by the author for the analysis and evaluation of the performance of various sign adaptive filters proposed by the author and other researchers. In particular, the same framework is used for steady-state analysis, tracking analysis, and transient analysis of various sign adaptive filters. In other words, the energy conservation framework stands out as a common theme that runs throughout the treatment of the performance of sign adaptive filters.

One of the features of the energy conservation approach employed for the evaluation of the performance of sign adaptive filters is that it allows for the evaluation of steady-state and tracking results without requiring a preliminary transient analysis.

An example of the unified application of the energy conservation framework in the aforementioned publications in this section is presented in Appendix B.

In [46], some insights were provided on the convergence and steady-state behaviors of the SLMF, SSLMF, Normalized Sign-Error Least Mean Fourth (NSLMF), and Normalized Sign-Sign Least Mean Fourth (NSSLMF) algorithms for both cases of real- and complex-valued data. In [47], the performance of the SRLMF and LMF algorithms is investigated in an adaptive channel equalization scenario. In [48], a novel cascaded 4-stage adaptive noise canceller is proposed for the removal of four artifacts present in the ElectroCardioGram (ECG) signal. In [49], some comments were reported on the adaptive noise cancelers implemented using the SLMF, SSLMF, and their variant algorithms. There were no new expressions derived in [46]–[49]. Therefore, there was no need for the application of the energy conservation framework in these publications.

1.8 Coherence/Consistency

The coherence or consistency in the publications listed in the author's original contributions can be demonstrated by the following facts:

1. The filter coefficients/weights update equations of various sign adaptive algorithms for real- and complex-valued data analyzed and evaluated in the publications listed in the author's original contributions are shown in Tables 1.1 and 1.2, respectively.

   These sign adaptive algorithms are derived from the traditional LMS, LMF, LMMN, Normalized Least Mean Square (NLMS), NLMF, and NLMMN algorithms by clipping either the input data or the estimation error, or both, which is accomplished by the application of signum function. Thus, the signum function is commonly encountered throughout the treatment of the performance of sign adaptive filters.

Table 1.1: Filter coefficients/weights update equations of various sign adaptive algorithms for real-valued data.

Algorithm	Filter Coefficients/Weights Update Equation
SSLMS [40]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} \mathrm{sign}[e_i]$
SRLMF [32]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3$
SRLMMN [34]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i[\delta + (1-\delta)e_i^2]$
NSLMS [44], [45]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\, \mathbf{u}_i^{\mathrm{T}} \mathrm{sign}[e_i]$
NSRLMS [43]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i$
NSRLMF [35], [36]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3$
NSRLMMN [37], [38]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i[\delta + (1-\delta)e_i^2]$

Table 1.2: Filter coefficients/weights update equations of various sign adaptive algorithms for complex-valued data.

Algorithm	Filter Coefficients/Weights Update Equation
SRLMS [39]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \, \mathrm{csgn}[\mathbf{u}_i]^* e_i$
SSLMS [41]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \, \mathrm{csgn}[\mathbf{u}_i]^* \mathrm{csgn}[e_i]$
SRLMF [33]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \, \mathrm{csgn}[\mathbf{u}_i]^* e_i \lvert e_i \rvert^2$
NSLMS [44], [45]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i \, \mathbf{u}_i^* \mathrm{csgn}[e_i]$
NSRLMS [42]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i \, \mathrm{csgn}[\mathbf{u}_i]^* e_i$

In Tables 1.1 and 1.2, $\mathbf{w}_i$ is the updated filter weight vector at iteration $i \geq 0$, μ is the fixed step-size, μ_i is the variable step-size depending on the normalization used, $\mathbf{u}_i$ is the regressor vector, $e_i = d_i - y_i$ is the estimation error signal, d_i is the desired signal, y_i is the adaptive filter output, δ is the mixing parameter ranging between $0 \leq \delta \leq 1$, $\mathrm{sign}(.)$ denotes the sign of its argument, and $\mathrm{csgn}(.)$ denotes the complex sign of its argument. The definition of the signum function for real-valued data is given by:

$$\mathrm{sign}[x] = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases} \tag{1.1}$$

The definition of the signum function for complex-valued data is given by:

$$\mathrm{csgn}[x] = \begin{cases} -1, & \text{if } \Re[x] < 0 \text{ or } (\Re[x] = 0 \text{ and } \Im[x] < 0), \\ 0, & \text{if } \Re[x] = \Im[x] = 0, \\ 1, & \text{if } \Re[x] > 0 \text{ or } (\Re[x] = 0 \text{ and } \Im[x] > 0). \end{cases} \tag{1.2}$$

2. While the performance of different adaptive filters has been studied separately in the open literature, the framework adopted in this thesis applies uniformly across various sign adaptive filters analyzed and evaluated in the publications listed in the author's original contributions.

   An advantage of applying energy conservation approach is that the tracking results of a particular sign adaptive filter can be obtained by mere inspection from its steady-state results as there are only minor differences. An example of this fact can be observed by comparing the expressions (B.1) and (B.4) in Appendix B for the steady-state Mean Square Error (MSE) and tracking MSE of the SRLMMN algorithm, respectively. The only difference between these two expressions is the presence of the term $\mu^{-1}\mathrm{Tr}(\mathbf{Q})$ in the tracking MSE of the SRLMMN algorithm.

1.9 Significance

Three of the novel sign adaptive algorithms proposed, analyzed, and evaluated by the author, namely SRLMF, SRLMMN, and NSRLMF have been successfully employed in applications ranging from power quality improvement in the distribution system and multiple artifacts reduction in the physiological signals. The significance of employing the above three novel algorithms in the aforementioned applications is outlined below:

1. In [53], the SRLMMN algorithm-based [34] control technique is successfully applied by other researchers in the real-time implementation of active shunt compensator for power quality improvement in the distribution system. The SRLMMN algorithm-based control technique has proven itself to be highly efficient in this particular application by offering fast convergence, less steady-state error, low total harmonic distortion, and less computation complexity when compared with the Recursive Least-Squares (RLS) and Variable Step-size Least Mean Square (VSLMS) algorithms [53].

2. The motivation for employing sign adaptive algorithms in different applications such as artifacts removal from various physiological signals such as ECG, ElectroEncephaloGram (EEG), etc. is due to their simplicity of implementation [48], [54], [55].

3. In [48], the SRLMF and SRLMMN algorithms were employed by the author in a novel cascaded 4-stage adaptive noise canceller for the removal of four artifacts present in the ECG signal, namely baseline wander, motion artifacts, muscle artifacts, and 60 Hz Power Line Interference (PLI).

4. In [54], adaptive algorithms such as NSLMS [44], [45], NSRLMS [43], and their variants were employed by other researchers for removing various artifacts from ECG signals.

5. In [55], the variants of the SRLMF algorithm [32] such as the NSRLMF [35], [36] and Block-Based Normalized Sign Regressor Least Mean Fourth (BBNSRLMF) algorithms were employed by other researchers for brain signal enhancement in remote health monitoring applications.

6. In [56], [57], the NSRLMF algorithm is successfully employed by other researchers for power quality improvement in wind-solar based distributed generation system. The NSRLMF algorithm is shown to outperform the LMF algorithm by displaying enhanced dynamic response amidst sudden system variations [56], [57]. It should be noted that the authors in [56] published their expanded work in [57] at the time of making minor amendments to my thesis.

1.10 Contributions

The contributions of this thesis are briefly outlined below and will be discussed in detail in the subsequent chapters:

1. The various sign adaptive algorithms analyzed and evaluated in this thesis can be classified into two categories, namely non-normalized and normalized. Each of this category can be further classified into two subcategories, namely real- and complex-valued data. This classification is depicted in Figure 1.5.

2. The four novel algorithms proposed, analyzed, and evaluated in this thesis include SRLMF [32], [33], SRLMMN [34], NSRLMF [35], [36], and NSRLMMN [37], [38]. The performance of the latter three algorithms has been analyzed and evaluated for real-valued data only. While the performance of the SRLMF algorithm has been analyzed and evaluated for both cases of real- and complex-valued data.

3. The other four algorithms analyzed and evaluated in this thesis include SRLMS [39], SSLMS [40], [41], NSRLMS [42], [43], and NSLMS [44], [45]. The performance of the latter three algorithms has been analyzed and evaluated for both cases of real- and complex-valued data. While the performance of the SRLMS algorithm has been analyzed and evaluated for complex-valued data only.

4. Finally, some misconceptions among biomedical signal processing researchers concerning the implementation of adaptive noise cancelers using the SLMF, SSLMF, and their variant algorithms were clarified in [46], [49].

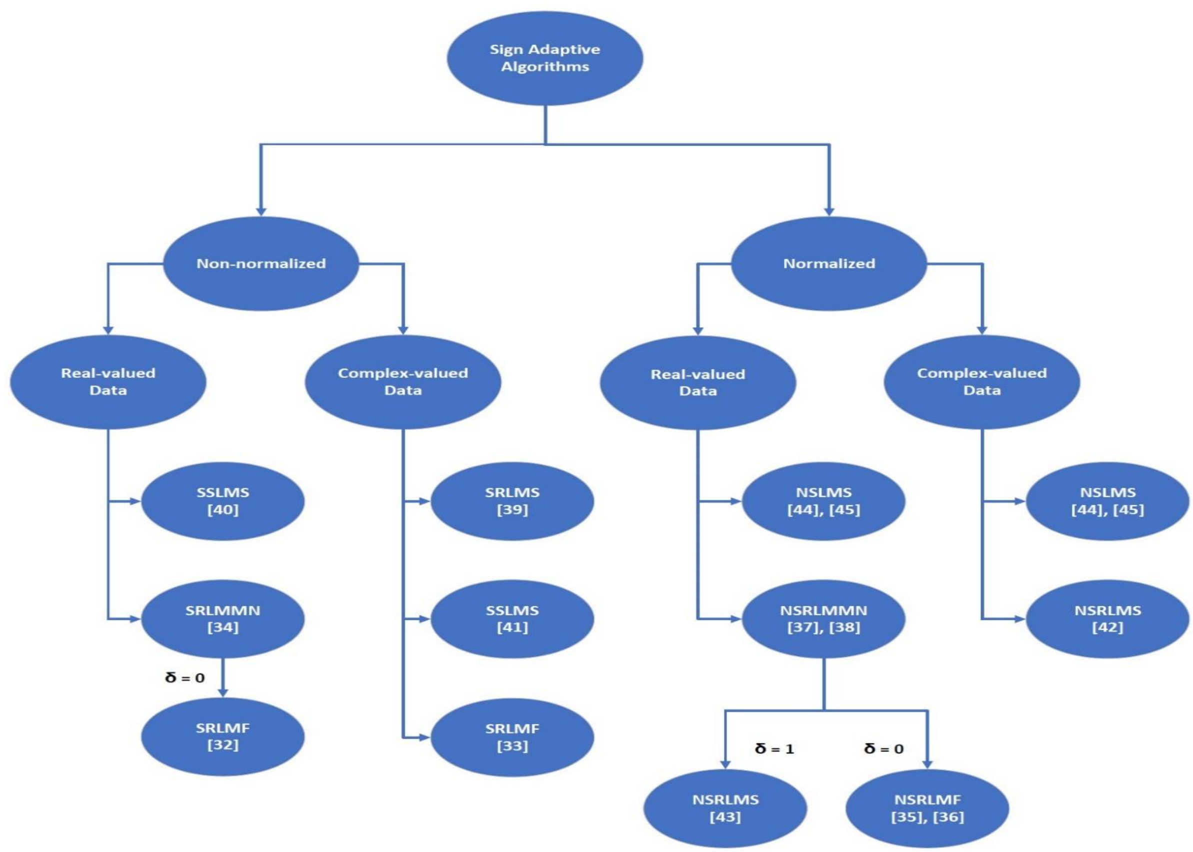

Figure 1.5: Family of the sign adaptive algorithms.

2 The SRLMF Algorithm

2.1 Introduction

The Sign Regressor Least Mean Fourth (SRLMF) algorithm is based on the clipping of the input data, which is also called as the regression data. The SRLMF algorithm belongs to the family of the Least Mean Fourth (LMF) algorithm. The only difference in the filter weights update equations of these two algorithms is the application of the signum function on the input data of the SRLMF algorithm.

The filter weights update equations of the SRLMF algorithm for real- and complex-valued data are given by (2.1) and (2.2), respectively [32], [33]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \qquad (2.1)$$

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \,\mathrm{csgn}[\mathbf{u}_i]^{*} e_i |e_i|^2, \qquad (2.2)$$

where $\mathbf{w}_i$ is the updated filter weight vector at iteration $i \geq 0$, μ is the step-size, $\mathbf{u}_i$ is the regressor vector, $e_i = d_i - y_i$ is the estimation error signal, d_i is the desired signal, y_i is the adaptive filter output, $\mathrm{sign}(.)$ denotes the sign of its argument, $\mathrm{csgn}(.)$ denotes the complex sign of its argument, and the definitions of the signum function for real- and complex-valued data are given by (1.1) and (1.2), respectively.

2.2 Background

The Sign Regressor Least Mean Square (SRLMS) algorithm, which is the counterpart of the SRLMF algorithm has been studied extensively in the open literature. However, there were no efforts made to study the performance evaluation of the SRLMF algorithm until it was proposed, analyzed, and evaluated in [32], [33].

The motivation to introduce the sign regressor term in the SRLMF algorithm is to achieve reduced computational complexity compared to the LMF algorithm. However, the convergence performance of the SRLMF algorithm is slower than the LMF algorithm but better than the SRLMS algorithm.

The advantage of employing the SRLMF algorithm in various applications is its computational simplicity. However, the simplification in computations for the SRLMF algorithm comes at the expense of slower convergence. The slow convergence in the performance of the SRLMF algorithm is because of the clipping effect of the signum function on the input data.

2.3 Contributions/Published Manuscripts

The three published papers on the performance evaluation of the SRLMF [32], [33], [47] algorithm for real- and complex-valued data are as follows:

[P1] M. M. U. Faiz, A. Zerguine, and A. Zidouri, "Analysis of the sign regressor least mean fourth adaptive algorithm," ***EURASIP Jour. on Advances in Signal Processing,*** **vol. 2011, art. no. 373205, pp. 1–12, Jan. 2011, DOI: https://doi.org/10.1155/2011/373205**

A novel adaptive algorithm called the SRLMF algorithm was proposed, analyzed, and evaluated for the case of real-valued data in [32]. The expressions for the steady-state Mean Square Error (MSE) $\varphi = \mathrm{E}[{e_i}^2]$ of the SRLMF algorithm were derived for both smaller and larger step-sizes and are given by (2.3) and (2.4), respectively [32]:

$$\varphi = \frac{\sqrt{2}\mu\xi_v^6 \mathrm{Tr}(\mathbf{R})}{6\sigma_v^2\sqrt{\pi\sigma_u^2}} + \sigma_v^2, \tag{2.3}$$

$$\varphi = \frac{\sqrt{2}\mu\xi_v^6 \mathrm{Tr}(\mathbf{R})}{6\sigma_v^2\sqrt{\pi\sigma_u^2} - 15\sqrt{2}\mu\xi_v^4\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{2.4}$$

Also, the expressions for the tracking MSE φ' of the SRLMF algorithm were derived for both smaller and larger step-sizes and are given by (2.5) and (2.6), respectively [32]:

$$\varphi' = \frac{\sqrt{2}\mu\xi_v^6 \mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{6\sigma_v^2\sqrt{\pi\sigma_u^2}} + \sigma_v^2, \tag{2.5}$$

$$\varphi' = \frac{\sqrt{2}\mu\xi_v^6 \mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{6\sigma_v^2\sqrt{\pi\sigma_u^2} - 15\sqrt{2}\mu\xi_v^4\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{2.6}$$

In addition, the expressions for the optimum step-size μ_{opt} of the SRLMF algorithm were also derived for both smaller and larger step-sizes and are given by (2.7) and (2.8), respectively [32]:

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\sqrt{\pi\sigma_u^2}\mathrm{Tr}(\mathbf{Q})}{\sqrt{2}\xi_v^6\mathrm{Tr}(\mathbf{R})}}, \tag{2.7}$$

$$\mu_{\mathrm{opt}} = \sqrt{\mathrm{Tr}(\mathbf{Q})\left[\frac{225\left(\xi_v^4\right)^2\mathrm{Tr}(\mathbf{Q})}{36\left(\sigma_v^2\right)^2\left(\xi_v^6\right)^2} + \frac{\sqrt{\pi\sigma_u^2}}{\sqrt{2}\xi_v^6\mathrm{Tr}(\mathbf{R})}\right]} - \frac{15\xi_v^4\mathrm{Tr}(\mathbf{Q})}{6\sigma_v^2\xi_v^6}. \tag{2.8}$$

A sufficient condition for the convergence in the mean of the SRLMF algorithm is also derived and is given by (2.9) [32]:

$$0 < \mu < \frac{\sqrt{2\pi\sigma_u^2}}{3\lambda_{\max}\sigma_e^2}. \tag{2.9}$$

In (2.3) to (2.9), $\sigma_v^2 = \mathrm{E}[{v_i}^2]$ is the noise variance, $\xi_v^4 = \mathrm{E}[{v_i}^4]$ and $\xi_v^6 = \mathrm{E}[{v_i}^6]$ are the fourth and sixth-order moments of the noise sequence v_i, respectively, $\sigma_u^2 = \mathrm{E}[{\mathbf{u}_i}^2]$ is the regressor variance, $\mathrm{Tr}(\mathbf{R})$ is the trace of the regressor covariance matrix $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i]$, $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$ of the noise sequence $\mathbf{q}_i$, $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{R}$, and σ_e^2 is the estimation error variance.

Moreover, the weighted variance relation has been extended in order to derive expressions for the MSE and Mean Square Deviation (MSD) of the SRLMF algorithm during the transient phase [32].

The convergence performance of the SRLMF and LMF algorithms is found to be almost identical for real-valued data in a uniform noise environment with an SNR of 10 dB as shown in Figure 1 [P1].

The steady-state MSE expressions of the SRLMF algorithm for both smaller and larger step-sizes given by (2.3) and (2.4), respectively, are compared with the simulation results in Figure 2 [P1]. In this Figure, the noise variance is fixed at $\sigma_v^2 = 0.001$, the adaptive filter length is fixed at $M = 5$, the step-size is varying from $\mu = 1e-3$ to $1e-1$, the length of white Gaussian regressors and white Gaussian noise is fixed at $N = 1e6$, the number of iterations are fixed at $L = 100$, and the coefficients of an unknown system identification setup are fixed at $\mathbf{w}_o = [0.227\ 0.460\ 0.688\ 0.460\ 0.227]^T$. The theoretical curves appear to be overlapping in Figure 2 [P1]. However, a zoom into the region around $\mu = 0.05$ in Figure 2 [P1] reveals that these two theoretical curves, although extremely close to each other, do not overlap as shown in Figure 2.1 below.

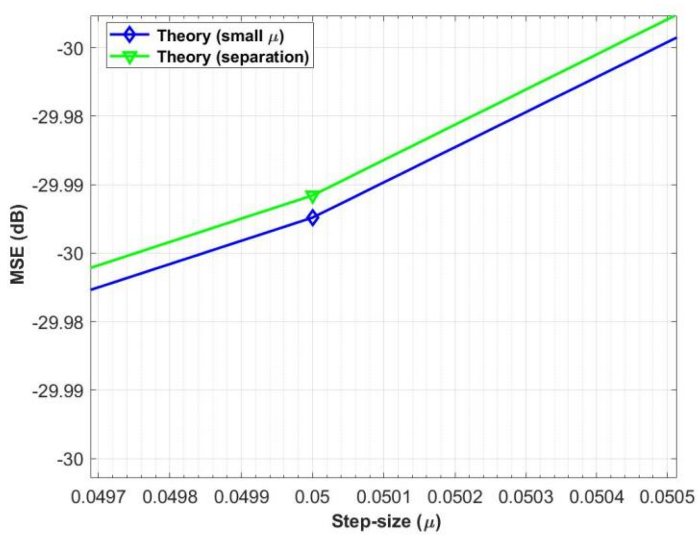

Figure 2.1: Comparison of the steady-state MSE expressions of the SRLMF algorithm using white Gaussian regressors - a closer look.

In Figure 2 [P1], it is observed that the simulations results vary in a zigzag manner with respect to the theoretical results. Such a behavior of the simulation results is because of the insufficient number of iterations. To prove our point, we have carried out the simulations for the number of iterations fixed at $L = 1000$ as shown in Figure 2.2 below.

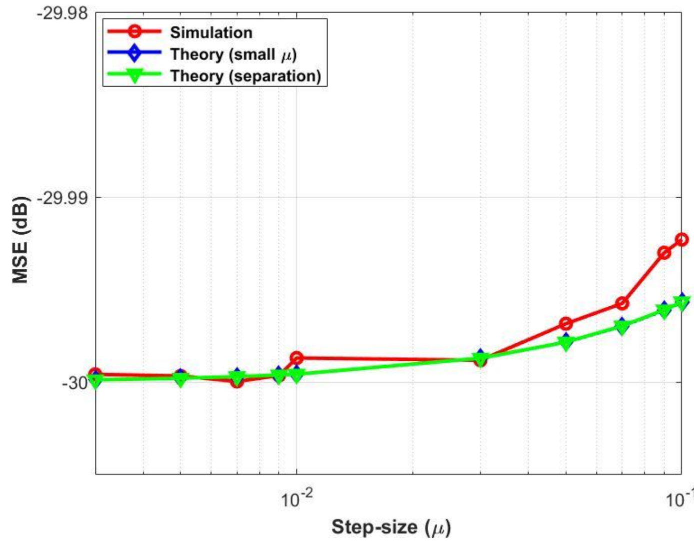

Figure 2.2: Comparison of the steady-state MSE expressions of the SRLMF algorithm using white Gaussian regressors.

In Figures 3 and 4 [P1], it is observed that the simulations results are not in a very good match with the theoretical results. This is again because of the insufficient number of iterations. To prove our point, we have carried out the simulations for the number of iterations fixed at $L = 1000$ as shown in Figures 2.3 and 2.4 below, which are much better than Figures 3 and 4 [P1], respectively.

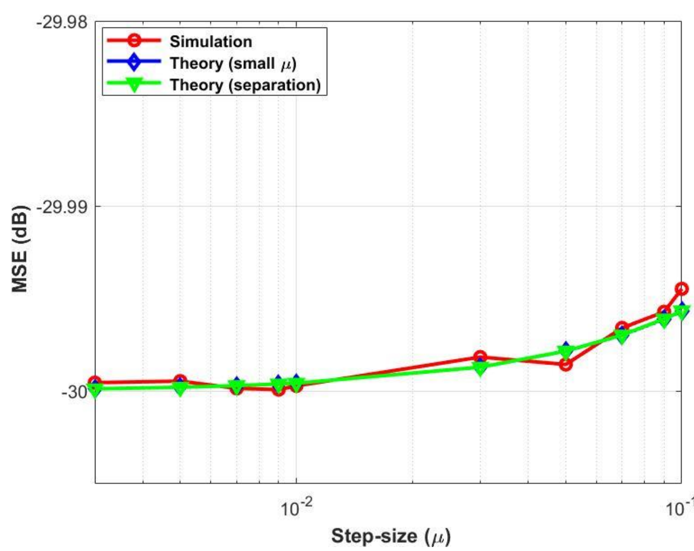

Figure 2.3: Comparison of the steady-state MSE expressions of the SRLMF algorithm using correlated Gaussian regressors.

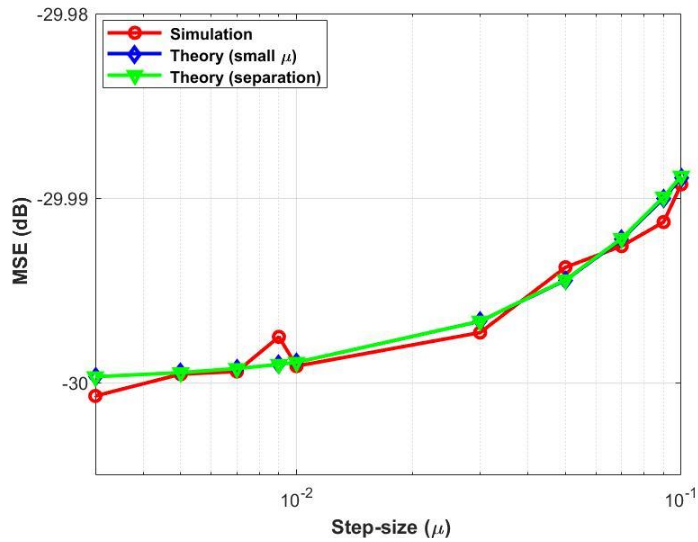

Figure 2.4: Comparison of the steady-state MSE expressions of the SRLMF algorithm using Gaussian regressors with an eigenvalue spread = 5.

[P2] M. M. U. Faiz and A. Zerguine, "Analysis of the complex sign regressor least mean fourth adaptive algorithm," ***in Proc. of the 2nd IEEE Int. Conf. on Signal and Image Processing Applications (ICSIPA 2011),*** **Kuala Lumpur, Malaysia, pp. 553–555, Nov. 2011, DOI: https://doi.org/10.1109/ICSIPA.2011.6144115**

After comprehensively analyzing and evaluating the performance of the SRLMF algorithm for the case of real-valued data in [32], the performance of the SRLMF algorithm was further investigated in an adaptive channel equalization scenario for both cases of real- and complex-valued data in [47]. This provided the motivation to analyze and evaluate the performance of the SRLMF algorithm for the case of complex-valued data in [33].

The expressions for the steady-state MSE $\varphi = \mathrm{E}[|e_i|^2]$ of the complex SRLMF algorithm were derived for both smaller and larger step-sizes and are given by (2.10) and (2.11), respectively [33]:

$$\varphi = \frac{\mu\xi_v^6\mathrm{Tr}(\mathbf{R})}{\sigma_v^2\sqrt{\pi\sigma_u^2}} + \sigma_v^2, \tag{2.10}$$

$$\varphi = \frac{\mu\xi_v^6\mathrm{Tr}(\mathbf{R})}{\sigma_v^2\sqrt{\pi\sigma_u^2} - 9\mu\xi_v^4\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{2.11}$$

Also, the expressions for the tracking MSE φ' of the complex SRLMF algorithm were derived for both smaller and larger step-sizes and are given by (2.12) and (2.13), respectively [33]:

$$\varphi' = \frac{4\mu\xi_v^6\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{4\sigma_v^2\sqrt{\pi\sigma_u^2}} + \sigma_v^2, \tag{2.12}$$

$$\varphi' = \frac{4\mu\xi_v^6 \mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{4\sigma_v^2\sqrt{\pi\sigma_u^2} - 36\mu\xi_v^4\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{2.13}$$

In addition, the expressions for the optimum step-size μ_{opt} of the complex SRLMF algorithm were also derived for both smaller and larger step-sizes and are given by (2.14) and (2.15), respectively [33]:

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\sqrt{\pi\sigma_u^2}\mathrm{Tr}(\mathbf{Q})}{4\xi_v^6\mathrm{Tr}(\mathbf{R})}}, \tag{2.14}$$

$$\mu_{\mathrm{opt}} = \sqrt{\mathrm{Tr}(\mathbf{Q})\left[\frac{81(\xi_v^4)^2\mathrm{Tr}(\mathbf{Q})}{16(\sigma_v^2)^2(\xi_v^6)^2} + \frac{\sqrt{\pi\sigma_u^2}}{4\xi_v^6\mathrm{Tr}(\mathbf{R})}\right]} - \frac{9\xi_v^4\mathrm{Tr}(\mathbf{Q})}{4\sigma_v^2\xi_v^6}. \tag{2.15}$$

In (2.10) to (2.15), $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ is the noise variance, $\xi_v^4 = \mathrm{E}[|v_i|^4]$ and $\xi_v^6 = \mathrm{E}[|v_i|^6]$ are the fourth and sixth-order moments of the noise sequence v_i, respectively, $\sigma_u^2 = \mathrm{E}[|\mathbf{u}_i|^2]$ is the regressor variance, $\mathrm{Tr}(\mathbf{R})$ is the trace of the regressor covariance matrix $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^*\mathbf{u}_i]$, and $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^*]$ of the noise sequence $\mathbf{q}_i$.

It is interesting to note that the expressions for the steady-state MSE, tracking MSE, and optimum step-size of the SRLMF algorithm for both real- and complex-valued data cases given from (2.3) to (2.8) and from (2.10) to (2.15) are found to be identical except for a scaling factor [32], [33]. Moreover, it is shown that the simulation results are in a good match with the analytical results for white Gaussian regressors.

Finally, a comparison between the convergence performance of the SRLMF and LMF algorithms indicates slower convergence of the SRLMF algorithm for complex-valued data in a uniform noise environment with an SNR of 10 dB.

[P3] M. M. U. Faiz and A. Zerguine, "Adaptive channel equalization using the sign regressor least mean fourth algorithm," ***in Proc. of the 1st IEEE Saudi Int. Electronics, Communications and Photonics Conf. (SIECPC 2011),*** **Riyadh, Saudi Arabia, pp. 1–4, Apr. 2011, DOI: https://doi.org/10.1109/SIECPC.2011.5876986**

The performance of the SRLMF and LMF algorithms was investigated when deployed in two types of adaptive channel equalizers, namely adaptive linear equalizer and adaptive decision feedback equalizer. The Bit Error Rate (BER) and MSE behaviors of the SRLMF and LMF algorithms were examined in both the above equalizers for Binary Phase Shift Keying (BPSK) and Quadrature Phase Shift Keying (QPSK) data in an Additive White Gaussian Noise (AWGN) and uniform noise environments, respectively. The filter weights update equations of the SRLMF algorithm for real- and complex-valued data given by (2.1) and (2.2) were used to handle BPSK and QPSK data, respectively [47].

For a given constellation, the BER performance of the SRLMF and LMF algorithms is similar for lower Signal to Noise Ratios (SNR's). As the SNR increases, an enhancement in the BER performance of the SRLMF algorithm is obtained over the LMF algorithm. However, the BER performance of the SRLMF and LMF algorithms is again similar for higher SNR's. Also, for a fixed value of SNR, it was observed that the probability of error increases as the order of the constellation increases [47].

Moreover, it was shown that the convergence performance of the SRLMF algorithm degrades compared to the LMF algorithm in both the aforementioned equalizers for both BPSK and QPSK data in a uniform noise environment with an SNR of 20 dB. Also, it was observed that the MSE increases as the order of the constellation increases [47].

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Research Article

Analysis of the Sign Regressor Least Mean Fourth Adaptive Algorithm

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A novel algorithm, called the signed regressor least mean fourth (SRLMF) adaptive algorithm, that reduces the computational cost and complexity while maintaining good performance is presented. Expressions are derived for the steady-state excess-mean-square error (EMSE) of the SRLMF algorithm in a stationary environment. A sufficient condition for the convergence in the mean of the SRLMF algorithm is derived. Also, expressions are obtained for the tracking EMSE of the SRLMF algorithm in a nonstationary environment, and consequently an optimum value of the step-size is obtained. Moreover, the weighted variance relation has been extended in order to derive expressions for the mean-square error (MSE) and the mean-square deviation (MSD) of the proposed algorithm during the transient phase. Computer simulations are carried out to corroborate the theoretical findings. It is shown that there is a good match between the theoretical and simulated results. It is also shown that the SRLMF algorithm has no performance degradation when compared with the least mean fourth (LMF) algorithm. The results in this study emphasize the usefulness of this algorithm in applications requiring reduced implementation costs for which the LMF algorithm is too complex.

1. Introduction

Reduction in complexity of the least mean square (LMS) algorithm has always received attention in the area of adaptive filtering [1–3]. This reduction is usually done by clipping either the estimation error or the input data, or both to reduce the number of multiplications necessary at each algorithm iteration. The algorithm based on clipping of the estimation error is known as the sign error or more commonly the sign algorithm (SA) [4–8], the algorithm based on clipping of the input data is known as the sign regressor algorithm (SRA) [9–12], and the algorithm based on clipping of both the estimation error and the input data is known as the sign sign algorithm (SSA) [13, 14]. These algorithms result in a performance loss when compared with the conventional LMS algorithm [9, 10]. However, significant reduction in computational cost and simplified hardware implementation can justify this poor performance in applications requiring reduced implementation costs [15, 16].

The behavior of the SRA algorithm depends on the input data. It is shown in [11] that for some inputs, the LMS algorithm is stable while the SRA algorithm is unstable. This is a drawback of the SRA algorithm when compared with the SA algorithm since the latter is more stable than the LMS algorithm [4, 16]. The SRA algorithm is always stable when the input data is Gaussian as in the case of speech processing. Also, the performance of the SRA algorithm is superior to that of the SA algorithm for Gaussian input data. It is shown in [10] that the SRA algorithm is much faster than the SA algorithm in achieving the desired steady-state mean-square error for white Gaussian data. Theoretical studies of the SRA algorithm with correlated Gaussian data in both stationary and nonstationary environments are found in [12].

The convergence rate and the steady-state mean-square error of the SRA algorithm is only slightly inferior to those of the LMS algorithm for the same parameter setting. In [10], the convergence rate of the SRA algorithm is compared with that of the LMS algorithm to show that the SRA algorithm converges slower than the LMS algorithm by a factor of $2/\pi$ for the same steady-state mean-square error.

It is shown in [17] that the SRA algorithm exhibits significantly higher robustness against the impulse noise than the LMS algorithm.

The above-mentioned advantages motivate us to analyze the proposed sign regressor least mean fourth (SRLMF) adaptive algorithm. In this paper, the mean-square analysis, the convergence analysis, the tracking analysis, and the transient analysis of the SRLMF algorithm are carried out. The framework used in this work relies on energy conservation arguments [18]. Expressions are evaluated for the steady-state excess-mean-square error (EMSE) of the SRLMF algorithm in a stationary environment. A condition for the convergence of the mean behavior of the SRLMF algorithm is also derived. Also, expressions for the tracking EMSE in a nonstationary environment are presented. An optimum value of the step-size μ is also evaluated. Moreover, an extension of the weighted variance relation is provided in order to derive expressions for the mean-square error (MSE) and the mean-square deviation (MSD) of the proposed algorithm during the transient phase. From the simulation results it is shown that both the SRLMF algorithm and the least mean fourth (LMF) algorithm [19] have a similar performance for the same steady-state EMSE. Moreover, the results show that the theoretical and simulated results are in good agreement.

The paper is organized as follows: following the Introduction is Section 2 where the proposed algorithm is developed, while the mean-square analysis of the proposed SRLMF algorithm is presented in Section 3. The convergence analysis of the proposed algorithm is presented in Section 4. Section 5 presents the tracking analysis of the proposed algorithm for random walk channels and as a by-product of this analysis the optimum value of step-size for these channels is derived. And Section 6 presents thoroughly the transient analysis of the proposed algorithm. The Computational Load is detailed in Section 7. To investigate the performance of the proposed algorithm, several simulation results for different scenarios are presented in Section 8. Finally, some conclusions are given in Section 9.

2. Algorithm Development

The SRLMF algorithm is based on clipping of the regression vector $\mathbf{u}_i$ (row vector). Consider now the adaptive filter, which updates its coefficients according to the following recursion [18]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^* \mathrm{g}[e_i], \quad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ (column vector) is the updated weight vector at time i, μ is the step-size, $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$, $\mathrm{g}[e_i]$ denotes some function of the estimation error signal given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}, \tag{2}$$

where d_i is the desired signal. When the data is real-valued and $\mathrm{g}[e_i] = e_i^3$, the general update form in (1) becomes

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}} e_i^3, \quad i \geq 0. \tag{3}$$

Now if

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{ \frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i_M}|} \right\}, \tag{4}$$

then the update form in (3) reduces to

$$\begin{aligned} \mathbf{w}_i &= \mathbf{w}_{i-1} + \mu\, \mathrm{diag}\left\{ \frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i_M}|} \right\} \mathbf{u}_i^{\mathrm{T}} e_i^3 \\ &= \mathbf{w}_{i-1} + \mu\, \mathrm{sign}\,[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \quad i \geq 0, \end{aligned} \tag{5}$$

where M is the filter length. The SRLMF algorithm update recursion in (5) can be regarded as a special case of the general update form in (3) for some matrix data nonlinearity that is implicitly defined by the following relation:

$$\mathrm{sign}\,[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}. \tag{6}$$

3. Mean-Square Analysis of the SRLMF Algorithm

We wil assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the stationary data model [18, 20–24].

(A.1) There exists an optimal weight vector $\mathbf{w}^o$ such that $d_i = \mathbf{u}_i \mathbf{w}^o + v_i$.

(A.2) The noise sequence v_i is independent and identically distributed (i.i.d.) with variance $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ and is independent of $\mathbf{u}_j$ for all i, j.

(A.3) The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.

(A.4) The regressor covariance matrix is $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^* \mathbf{u}_i] > \mathbf{0}$.

For any adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [18]:

$$\mu \mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 \mathrm{g}^2[e_i] \right] = 2\mathrm{E}[e_{a_i} \mathrm{g}[e_i]], \quad \text{as } i \longrightarrow \infty, \tag{7}$$

where

$$\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 \right] = \mathrm{E}\left[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}} \right], \tag{8}$$

$$e_i = e_{a_i} + v_i, \tag{9}$$

and $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error. Then $\mathrm{g}[e_i]$ becomes

$$\mathrm{g}[e_i] = e_i^3 = (e_{a_i} + v_i)\left[e_{a_i}^2 + v_i^2 + 2e_{a_i} v_i \right]. \tag{10}$$

By using the fact that e_{a_i} and v_i are independent, we reach at the following expression for the term $\mathrm{E}[e_{a_i} \mathrm{g}[e_i]]$:

$$\mathrm{E}[e_{a_i} \mathrm{g}[e_i]] = 3\sigma_v^2 \mathrm{E}\left[e_{a_i}^2 \right] + \mathrm{E}\left[e_{a_i}^4 \right]. \tag{11}$$

Ignoring third and higher-order terms of e_{a_i}, then (11) becomes

$$\mathrm{E}[e_{a_i}\mathrm{g}[e_i]] \approx 3\sigma_v^2 \mathrm{E}\left[e_{a_i}^2\right]. \quad (12)$$

To evaluate the term $\mathrm{E}[\|\mathbf{u}_i\|_{\mathrm{H}}^2 \mathrm{g}^2[e_i]]$, we start by noting that

$$\begin{aligned} \mathrm{g}^2[e_i] = {} & e_{a_i}^6 + 6e_{a_i}^5 v_i + 6e_{a_i} v_i^5 + 15e_{a_i}^4 v_i^2 + 15e_{a_i}^2 v_i^4 \\ & + 20e_{a_i}^3 v_i^3 + v_i^6. \end{aligned} \quad (13)$$

If we multiply $\mathrm{g}^2[e_i]$ by $\|\mathbf{u}_i\|_{\mathrm{H}}^2$ from the left, use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, and again ignoring third and higher-order terms of e_{a_i}, we obtain

$$\begin{aligned} \mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] & \approx 6\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i} v_i^5\right] + 15\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}^2 v_i^4\right] \\ & \quad + \mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 v_i^6\right] \\ & \approx 6\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}\right]\mathrm{E}\left[v_i^5\right] + 15\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}^2\right] \\ & \quad \times \mathrm{E}[v_i^4] + \mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2\right]\mathrm{E}\left[v_i^6\right] \\ & \approx 6\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}\right]\mathrm{E}\left[v_i^5\right] + 15\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}^2\right]\xi_v^4 \\ & \quad + \mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2\right]\xi_v^6, \end{aligned} \quad (14)$$

where $\xi_v^4 = \mathrm{E}[|v_i|^4]$, $\xi_v^6 = \mathrm{E}[|v_i|^6]$ denote the forth and sixth-order moments of v_i, respectively.

From Price's theorem [25] we have

$$\mathrm{E}[x\,\mathrm{sign}(y)] = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_y} \mathrm{E}[xy], \quad (15)$$

then

$$\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2\right] = \mathrm{E}\left[\mathbf{u}_i\,\mathrm{sign}\,[\mathbf{u}_i]^{\mathrm{T}}\right] = \sqrt{\frac{2}{\pi\sigma_u^2}}\,\mathrm{Tr}(\mathbf{R}). \quad (16)$$

Substituting (16) into (14) we get

$$\begin{aligned} \mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] & \approx 6\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}\right]\mathrm{E}\left[v_i^5\right] + 15\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}^2\right]\xi_v^4 \\ & \quad + \sqrt{\frac{2}{\pi\sigma_u^2}}\,\mathrm{Tr}(\mathbf{R})\xi_v^6. \end{aligned} \quad (17)$$

Substituting (12) and (17) into (7) we get

$$\begin{aligned} 6\sigma_v^2 \mathrm{E}\left[e_{a_i}^2\right] = {} & \mu\xi_v^6 \sqrt{\frac{2}{\pi\sigma_u^2}}\,\mathrm{Tr}(\mathbf{R}) + 15\mu\xi_v^4 \mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}^2\right] \\ & + 6\mu\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}\right]\mathrm{E}\left[v_i^5\right]. \end{aligned} \quad (18)$$

In order to simplify (18) and arrive at an expression for the steady-state EMSE $\zeta = \mathrm{E}[e_{a_i}^2]$, we consider two cases.

(1) Sufficiently Small Step-Sizes. Small step-sizes lead to small values of $\mathrm{E}[e_{a_i}^2]$ and e_{a_i} in steady-state. Therefore, for smaller values of μ, the last two terms in (18) can be ignored, the steady-state EMSE is given by

$$\zeta = \frac{\mu\xi_v^6}{6\sigma_v^2} \sqrt{\frac{2}{\pi\sigma_u^2}}\,\mathrm{Tr}(\mathbf{R}). \quad (19)$$

(2) Separation Principle. For larger values of μ, we resort to the separation assumption, namely, that at steady-state, $\|\mathbf{u}_i\|_{\mathrm{H}}^2$ is independent of e_{a_i}. In this case, the last term in (18) will be zero since e_{a_i} is zero mean, the steady-state EMSE can be shown to be

$$\zeta = \frac{\mu\xi_v^6 \sqrt{2/\pi\sigma_u^2}\,\mathrm{Tr}(\mathbf{R})}{\left(6\sigma_v^2 - 15\mu\xi_v^4 \sqrt{2/\pi\sigma_u^2}\,\mathrm{Tr}(\mathbf{R})\right)}. \quad (20)$$

4. Convergence Analysis of the SRLMF Algorithm

Convergence analysis of the SRLMF algorithm is much more complicated than that of the LMS algorithm due to existence of the higher order estimation error signal in the coefficient update recursion. We thus make the following assumptions along with (A.2) to make the analysis mathematically more tractable [19–24, 26]:

(A.5) d_i and $\mathbf{u}_i$ are zero-mean, wide-sense stationary, and jointly Gaussian random variables.

(A.6) The input pair $\{d_i, \mathbf{u}_i\}$ is independent of $\{d_j, \mathbf{u}_j\}$ for all i, j.

Subtracting both sides of (5) from $\mathbf{w}^o$ we get

$$\tilde{\mathbf{w}}_i = \tilde{\mathbf{w}}_{i-1} + \mu\,\mathrm{sign}\,[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \quad (21)$$

where $\tilde{\mathbf{w}}_i = \mathbf{w}^o - \mathbf{w}_i$. Taking expectations of both sides of (21) we obtain

$$\mathrm{E}[\tilde{\mathbf{w}}_i] = \mathrm{E}[\tilde{\mathbf{w}}_{i-1}] + \mu\mathrm{E}\left[\mathrm{sign}\,[\mathbf{u}_i]^{\mathrm{T}} e_i^3\right]. \quad (22)$$

Using Price's theorem [25], we can conclude that

$$\mathrm{E}\left[\mathrm{sign}\,[\mathbf{u}_i]^{\mathrm{T}} e_i^3\right] = \sqrt{\frac{2}{\pi\sigma_u^2}}\,\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} e_i^3\right]. \quad (23)$$

Substituting (23) into (22) we get

$$\mathrm{E}[\tilde{\mathbf{w}}_i] = \mathrm{E}[\tilde{\mathbf{w}}_{i-1}] + \mu\sqrt{\frac{2}{\pi\sigma_u^2}}\,\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} e_i^3\right]. \quad (24)$$

The expectation $\mathrm{E}[\mathbf{u}_i^{\mathrm{T}} e_i^3]$ can be simplified using the fact that for zero-mean and jointly Gaussian random variables x_1 and x_2,

$$\mathrm{E}[x_1 x_2^3] = 3\mathrm{E}[x_1 x_2]\mathrm{E}[x_2^2]. \quad (25)$$

Thus, using (25) in conjunction with (A.5), it follows that

$$\begin{aligned}\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} e_i^3\right] &= \mathrm{E}\left[\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} e_i^3 \mid \tilde{\mathbf{w}}_{i-1}\right]\right] \\ &= 3\mathrm{E}[\mathrm{E}[e_i^2 \mid \tilde{\mathbf{w}}_{i-1}]]\mathrm{E}\left[\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} e_i \mid \tilde{\mathbf{w}}_{i-1}\right]\right] \\ &= 3\mathrm{E}\left[\sigma^2_{e|\tilde{\mathbf{w}}_{i-1}}\right]\mathrm{E}\left[\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} e_i \mid \tilde{\mathbf{w}}_{i-1}\right]\right],\end{aligned} \tag{26}$$

where

$$\begin{aligned}\mathrm{E}\left[\sigma^2_{e|\tilde{\mathbf{w}}_{i-1}}\right] &= \sigma_e^2 - \mathrm{var}\{\mathrm{E}[e_i \mid \tilde{\mathbf{w}}_{i-1}]\} \\ &= \sigma_e^2,\end{aligned} \tag{27}$$

and from (9)

$$\begin{aligned}\mathrm{E}\left[\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} e_i \mid \tilde{\mathbf{w}}_{i-1}\right]\right] &= \mathrm{E}\left[\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} (v_i + \mathbf{u}_i\tilde{\mathbf{w}}_{i-1}) \mid \tilde{\mathbf{w}}_{i-1}\right]\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_i\tilde{\mathbf{w}}_{i-1} \mid \tilde{\mathbf{w}}_{i-1}\right]\right] \\ &= \mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_i\tilde{\mathbf{w}}_{i-1}\right] \\ &= \mathbf{R}\mathrm{E}[\tilde{\mathbf{w}}_{i-1}].\end{aligned} \tag{28}$$

Substituting (27) and (28) in (26) yields

$$\mathrm{E}\left[\mathbf{u}_i^{\mathrm{T}} e_i^3\right] = 3\sigma_e^2 \mathbf{R}\mathrm{E}[\tilde{\mathbf{w}}_{i-1}]. \tag{29}$$

Substituting (29) into (24) we get

$$\begin{aligned}\mathrm{E}[\tilde{\mathbf{w}}_i] &= \mathrm{E}[\tilde{\mathbf{w}}_{i-1}] + 3\mu\sqrt{\frac{2}{\pi\sigma_u^2}}\sigma_e^2\mathbf{R}\mathrm{E}[\tilde{\mathbf{w}}_{i-1}] \\ &= \left[\mathrm{I} + 3\mu\sqrt{\frac{2}{\pi\sigma_u^2}}\sigma_e^2\mathbf{R}\right]\mathrm{E}[\tilde{\mathbf{w}}_{i-1}].\end{aligned} \tag{30}$$

Ultimately, it is easy to show that the mean behavior of the weight vector, that is $\mathrm{E}[\mathbf{w}_i]$, converges to the optimal weight vector $\mathbf{w}^o$ if μ is bounded by:

$$0 < \mu < \frac{\sqrt{2\pi\sigma_u^2}}{3\lambda_{\max}\sigma_e^2}, \tag{31}$$

where $\lambda_{\max}$ represents the maximum eigenvalue of the regressor covariance matrix $\mathbf{R}$. Notice, that there exists the time-varying function σ_e^2 and the regressor variance σ_u^2 in the upper bound for μ. Since σ_e^2 is usually large at the beginning of adaptation processes, we can see that the convergence of the SRLMF algorithm strongly depends on the choice of initial conditions.

5. Tracking Analysis of the SRLMF Algorithm

Here, we assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the nonstationary data model [18].

(A.7) There exists a vector $\mathbf{w}_i^o$ such that $d_i = \mathbf{u}_i\mathbf{w}_i^o + v_i$.

(A.8) The weight vector varies according to the random-walk model $\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i$, and the sequence $\mathbf{q}_i$ is i.i.d. with covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.

(A.9) The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

In this case, the following variance relation holds [18]:

$$\mu\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 g^2[e_i]\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}[e_{a_i} g[e_i]], \quad \text{as } i \longrightarrow \infty. \tag{32}$$

Tracking results can be obtained by inspection from the mean-square results as there are only minor differences. Therefore, by substituting (12) and (17) into (32), we get

$$\begin{aligned}6\sigma_v^2\mathrm{E}\left[e_{a_i}^2\right] &= \mu^{-1}\mathrm{Tr}(\mathbf{Q}) + \mu\xi_v^6\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}) \\ &\quad + 15\mu\xi_v^4\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}^2\right] \\ &\quad + 6\mu\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}\right]\mathrm{E}\left[v_i^5\right].\end{aligned} \tag{33}$$

We again consider two cases for the evaluation of the tracking EMSE ζ of the SRLMF algorithm.

(1) Sufficiently Small Step-Sizes. Also, here, in this case we get

$$\zeta = \frac{\mu^{-1}\mathrm{Tr}(\mathbf{Q}) + \mu\xi_v^6\sqrt{2/\pi\sigma_u^2}\,\mathrm{Tr}(\mathbf{R})}{6\sigma_v^2}. \tag{34}$$

An optimum value of the step-size of the SRLMF algorithm is obtained by minimizing (34) with respect to μ. Setting the derivative of ζ with respect to μ equal to zero gives

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sqrt{2/\pi\sigma_u^2}\,\mathrm{Tr}(\mathbf{R})\xi_v^6}}. \tag{35}$$

(2) Separation Principle. Similarly here as it was done for the derivation of (20), we obtain the following:

$$\zeta = \frac{\mu^{-1}\mathrm{Tr}(\mathbf{Q}) + \mu\xi_v^6\sqrt{2/\pi\sigma_u^2}\,\mathrm{Tr}(\mathbf{R})}{\left(6\sigma_v^2 - 15\mu\xi_v^4\sqrt{2/\pi\sigma_u^2}\,\mathrm{Tr}(\mathbf{R})\right)}, \tag{36}$$

and eventually the optimum step-size of the SRLMF algorithm is given by

$$\begin{aligned}\mu_{\mathrm{opt}} = &\sqrt{\mathrm{Tr}(\mathbf{Q})\left[\frac{225(\xi_v^4)^2\,\mathrm{Tr}(\mathbf{Q})}{36(\sigma_v^2)^2(\xi_v^6)^2} + \frac{1}{\sqrt{2/\pi\sigma_u^2}\,\mathrm{Tr}(\mathbf{R})\xi_v^6}\right]} \\ &- \frac{15\xi_v^4}{6\sigma_v^2\xi_v^6}\mathrm{Tr}(\mathbf{Q}).\end{aligned} \tag{37}$$

6. Transient Analysis of the SRLMF Algorithm

Here, we will assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the conditions of the stationary data model described in Section 3.

6.1. Weighted Energy-Conservation Relation

Theorem 1. *For any adaptive filter of the form* (1), *any positive-definite Hermitian matrix* Σ, *and for any data* $\{d_i, \mathbf{u}_i\}$, *it holds that [18]:*

$$\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\|\tilde{\mathbf{w}}_i\|^2_{\Sigma} + \left|e^{\mathrm{H}\Sigma}_{a_i}\right|^2 = \|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma} + \left|e^{\mathrm{H}\Sigma}_{p_i}\right|^2, \tag{38}$$

where $e^{\mathrm{H}\Sigma}_{a_i}\mathbf{u}_i\mathrm{H}[\mathbf{u}_i]\Sigma\tilde{\mathbf{w}}_{i-1}$, $e^{\mathrm{H}\Sigma}_{p_i}\mathbf{u}_i\mathrm{H}[\mathbf{u}_i]\Sigma\tilde{\mathbf{w}}_i$, *and* $\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}} = \mathbf{u}_i(\mathrm{H}[\mathbf{u}_i]\Sigma\mathrm{H}[\mathbf{u}_i])\mathbf{u}_i^*$.

Proof. Let us consider the adaptive filter updates of the generic form given in (1). Subtracting both sides of (1) from $\mathbf{w}^o$, we get

$$\tilde{\mathbf{w}}_i = \tilde{\mathbf{w}}_{i-1} - \mu\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*\mathrm{g}[e_i]. \tag{39}$$

If we multiply both sides of (39) by $\mathbf{u}_i\mathrm{H}[\mathbf{u}_i]\Sigma$ from the left, we get

$$e^{\mathrm{H}\Sigma}_{p_i} = e^{\mathrm{H}\Sigma}_{a_i} - \mu\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}[e_i]. \tag{40}$$

Two cases can be considered here.

Case 1 ($\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}} = 0$). In this case, $\tilde{\mathbf{w}}_i = \tilde{\mathbf{w}}_{i-1}$ and $e^{\mathrm{H}\Sigma}_{a_i} = e^{\mathrm{H}\Sigma}_{p_i}$ so that $\|\tilde{\mathbf{w}}_i\|^2_{\Sigma} = \|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma}$ and $|e^{\mathrm{H}\Sigma}_{a_i}|^2 = |e^{\mathrm{H}\Sigma}_{p_i}|^2$.

Case 2 ($\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}} \neq 0$). In this case, we use (40) to solve for $\mathrm{g}[e_i]$,

$$\mathrm{g}[e_i] = \frac{1}{\mu\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}}\left(e^{\mathrm{H}\Sigma}_{a_i} - e^{\mathrm{H}\Sigma}_{p_i}\right). \tag{41}$$

Substituting (41) into (39), we get

$$\tilde{\mathbf{w}}_i = \tilde{\mathbf{w}}_{i-1} - \frac{\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*}{\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}}\left(e^{\mathrm{H}\Sigma}_{a_i} - e^{\mathrm{H}\Sigma}_{p_i}\right). \tag{42}$$

Expression (42) can be rearranged as

$$\tilde{\mathbf{w}}_i + \frac{\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*}{\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}}e^{\mathrm{H}\Sigma}_{a_i} = \tilde{\mathbf{w}}_{i-1} + \frac{\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*}{\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}}e^{\mathrm{H}\Sigma}_{p_i}. \tag{43}$$

Evaluating the energies of both sides of (43) results in

$$\left\|\tilde{\mathbf{w}}_i + \frac{\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*}{\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}}e^{\mathrm{H}\Sigma}_{a_i}\right\|^2_{\Sigma} = \left\|\tilde{\mathbf{w}}_{i-1} + \frac{\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*}{\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}}e^{\mathrm{H}\Sigma}_{p_i}\right\|^2_{\Sigma}. \tag{44}$$

After a straightforward calculation, the following weighted energy-conservation results:

$$\|\tilde{\mathbf{w}}_i\|^2_{\Sigma} + \frac{1}{\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}}\left|e^{\mathrm{H}\Sigma}_{a_i}\right|^2 = \|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma} + \frac{1}{\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}}\left|e^{\mathrm{H}\Sigma}_{p_i}\right|^2. \tag{45}$$

The weighted energy-conservation relation in (45) can also be written as

$$\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\|\tilde{\mathbf{w}}_i\|^2_{\Sigma} + \left|e^{\mathrm{H}\Sigma}_{a_i}\right|^2 = \|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma} + \left|e^{\mathrm{H}\Sigma}_{p_i}\right|^2. \tag{46}$$

6.2. Weighted Variance Relation. Here, the weighted variance relation presented in [18] has been extended in order to derive expressions for the MSE and the MSD of the SRLMF algorithm during the transient phase.

Theorem 2. *For any adaptive filter of the form* (1), *any positive-definite Hermitian matrix* Σ, *and for any data* $\{d_i, \mathbf{u}_i\}$, *it holds that*

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\Sigma}\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma}\right] + \mu^2\mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\left|\mathrm{g}[e_i]\right|^2\right] \\ &\quad -2\mu\,\mathrm{Re}\left(\mathrm{E}\left[e^{\mathrm{H}\Sigma*}_{a_i}\mathrm{g}[e_i]\right]\right), \quad \textit{as } i \longrightarrow \infty.\end{aligned} \tag{47}$$

Similarly, for real-valued data, the above weighted variance relation becomes

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\Sigma}\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma}\right] + \mu^2\mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}^2[e_i]\right] \\ &\quad -2\mu\mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \longrightarrow \infty.\end{aligned} \tag{48}$$

Proof. Squaring both sides of (40), we get

$$\left|e^{\mathrm{H}\Sigma}_{p_i}\right|^2 = \left|e^{\mathrm{H}\Sigma}_{a_i} - \mu\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}[e_i]\right|^2. \tag{49}$$

For compactness of notation let us omit the argument of g so that (49) looks like

$$\begin{aligned}\left|e^{\mathrm{H}\Sigma}_{p_i}\right|^2 &= \left|e^{\mathrm{H}\Sigma}_{a_i}\right|^2 + \mu^2\|\mathbf{u}_i\|^4_{\mathrm{H}\Sigma\mathrm{H}}|\mathrm{g}|^2 - \mu e^{\mathrm{H}\Sigma}_{a_i}\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}^* \\ &\quad - \mu e^{\mathrm{H}\Sigma*}_{a_i}\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}.\end{aligned} \tag{50}$$

Substituting (50) into (46), we get

$$\begin{aligned}\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\|\tilde{\mathbf{w}}_i\|^2_{\Sigma} &= \|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma} + \mu^2\|\mathbf{u}_i\|^4_{\mathrm{H}\Sigma\mathrm{H}}|\mathrm{g}|^2 \\ &\quad - \mu e^{\mathrm{H}\Sigma}_{a_i}\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}^* - \mu e^{\mathrm{H}\Sigma*}_{a_i}\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}.\end{aligned} \tag{51}$$

Dividing both sides of (51) by $\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}$ (of course here $\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}} \neq 0$) we get

$$\|\tilde{\mathbf{w}}_i\|^2_{\Sigma} = \|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma} + \mu^2\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}|\mathrm{g}|^2 - \mu e^{\mathrm{H}\Sigma}_{a_i}\mathrm{g}^* - \mu e^{\mathrm{H}\Sigma*}_{a_i}\mathrm{g}. \tag{52}$$

Taking expectations of both sides of (52), we obtain

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\Sigma}\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma}\right] + \mu^2\mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\left|\mathrm{g}[e_i]\right|^2\right] \\ &\quad - \mu\mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}\mathrm{g}[e_i]^* + e^{\mathrm{H}\Sigma*}_{a_i}\mathrm{g}[e_i]\right],\end{aligned} \tag{53}$$

or in the following format:

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\Sigma}\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma}\right] + \mu^2\mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\left|\mathrm{g}[e_i]\right|^2\right] \\ &\quad - 2\mu\,\mathrm{Re}\left(\mathrm{E}\left[e^{\mathrm{H}\Sigma*}_{a_i}\mathrm{g}[e_i]\right]\right), \quad \text{as } i \longrightarrow \infty.\end{aligned} \tag{54}$$

For real-valued data, the weighted variance relation in (54) becomes

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\Sigma}\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma}\right] + \mu^2\mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}^2[e_i]\right] \\ &\quad -2\mu\mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \longrightarrow \infty.\end{aligned} \tag{55}$$

The transient analysis of the class of filters in (1) is more challenging due to the presence of the error nonlinearity. Nevertheless, by using some approximations, the analysis can be carried out to provide some useful insights about the performance of the SRLMF algorithm.

To start, the expectations $\mathrm{E}[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}} \mathrm{g}^2[e_i]]$ and $\mathrm{E}[e^{\mathrm{H}\Sigma}_{a_i} \mathrm{g}[e_i]]$ are evaluated in the ensuing analysis in terms of the weighted norm of $\tilde{\mathbf{w}}_{i-1}$. Since these expectations are involved mathematically we will rely on the following assumption in order to facilitate their evaluation [18].

(A.10) The a priori estimation errors $\{e_{a_i}, e^{\mathrm{H}\Sigma}_{a_i}\}$ are jointly circular Gaussian.

Evaluation of $E[e^{H\Sigma}_{a_i} g[e_i]]$. From Price's theorem, if x and y are jointly Gaussian random variables that are independent from a third random variable z, then it holds that [25]:

$$\mathrm{E}[x\mathrm{g}(y+z)] = \frac{\mathrm{E}[xy]}{\mathrm{E}[y^2]}\mathrm{E}[y\mathrm{g}(y+z)]. \tag{56}$$

Applying this result to the term $\mathrm{E}[e^{\mathrm{H}\Sigma}_{a_i} \mathrm{g}[e_i]]$, and using (9), we get

$$\begin{aligned} \mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}\mathrm{g}[e_i]\right] &= \mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}\mathrm{g}[e_{a_i}+v_i]\right] \\ &= \mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}e_{a_i}\right]\left[\frac{\mathrm{E}[e_{a_i}\mathrm{g}[e_i]]}{\mathrm{E}\left[e^2_{a_i}\right]}\right]. \end{aligned} \tag{57}$$

In view of the assumption (A.10), the expectation $\mathrm{E}[e_{a_i}\mathrm{g}[e_i]]$ depends on e_{a_i} only through its second moment, $\mathrm{E}[e^2_{a_i}]$. Therefore, we can define the following function of $\mathrm{E}[e^2_{a_i}]$:

$$\mathcal{Z}_1 = \frac{\mathrm{E}[e_{a_i}\mathrm{g}[e_i]]}{\mathrm{E}\left[e^2_{a_i}\right]}. \tag{58}$$

For the SRLMF algorithm, $\mathrm{g}[e_i] = e_i^3$, therefore

$$\begin{aligned} \mathrm{E}[e_{a_i}\mathrm{g}[e_i]] &= \mathrm{E}\left[e_{a_i}(e_{a_i}+v_i)^3\right] \\ &= \mathrm{E}\left[e^4_{a_i} + 3e^3_{a_i}v_i + 3e^2_{a_i}v_i^2 + v_i^3 e_{a_i}\right]. \end{aligned} \tag{59}$$

Now since e_{a_i} and v_i are zero mean Gaussian and independent random variables with variances $\mathrm{E}[e^2_{a_i}]$ and σ_v^2, respectively, we obtain

$$\mathrm{E}[e_{a_i}\mathrm{g}[e_i]] = \mathrm{E}\left[e^4_{a_i}\right] + 3\sigma_v^2\mathrm{E}\left[e^2_{a_i}\right]. \tag{60}$$

By using the fact that for circular Gaussian e_{a_i} it holds that $\mathrm{E}[e^4_{a_i}] = 3\mathrm{E}[e^2_{a_i}]^2$, we get

$$\begin{aligned} \mathrm{E}[e_{a_i}\mathrm{g}[e_i]] &= 3\mathrm{E}\left[e^2_{a_i}\right]^2 + 3\sigma_v^2\mathrm{E}\left[e^2_{a_i}\right] \\ &= 3\mathrm{E}\left[e^2_{a_i}\right]\left[\mathrm{E}\left[e^2_{a_i}\right]+\sigma_v^2\right]. \end{aligned} \tag{61}$$

Substituting (61) into (58), we get

$$\mathcal{Z}_1 = 3\left[\mathrm{E}\left[e^2_{a_i}\right]+\sigma_v^2\right]. \tag{62}$$

The expression for $\mathcal{Z}_1$ is related to the desired term $\mathrm{E}[e^{\mathrm{H}\Sigma}_{a_i}\mathrm{g}[e_i]]$ through the equality

$$\mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}\mathrm{g}[e_i]\right] = \mathcal{Z}_1\mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}e_{a_i}\right]. \tag{63}$$

Evaluation of $E[\|\mathbf{u}_i\|^2_{H\Sigma H} g^2[e_i]]$. In order to facilitate the evaluation of the term $\mathrm{E}[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}^2[e_i]]$ we use the separation principle, namely, we assume that the filter is long enough so that the following assumption holds [18].

(A.11) $\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}$ is independent of e_i.

Therefore,

$$\mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}^2[e_i]\right] = \left(\mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\right]\right)\left(\mathrm{E}[\mathrm{g}^2[e_i]]\right). \tag{64}$$

Since e_{a_i} is Gaussian and independent of the noise, the expectation $\mathrm{E}[\mathrm{g}^2[e_i]]$ depends on e_{a_i} through its second moment only. Therefore, we can define the following function of $\mathrm{E}[e^2_{a_i}]$:

$$\mathcal{Z}_2 = \mathrm{E}[\mathrm{g}^2[e_i]]. \tag{65}$$

For the SRLMF algorithm, $\mathrm{g}[e_i] = e_i^3$. Since e_{a_i} and v_i are zero mean Gaussian and independent random variables with variances $\mathrm{E}[e^2_{a_i}]$ and σ_v^2, we have $\sigma_e^2 = \mathrm{E}[e_i^2] = \mathrm{E}[e^2_{a_i}] + \sigma_v^2$. Moreover from [18], $\mathrm{E}[e_i^6] = 15\sigma_e^6$.
Thus

$$\begin{aligned} \mathcal{Z}_2 &= \mathrm{E}\left[e_i^6\right] \\ &= 15\sigma_e^6 \\ &= 15\left(\sigma_e^2\right)^3 \\ &= 15\left(\mathrm{E}\left[e^2_{a_i}\right]+\sigma_v^2\right)^3 \\ &= 15\left(\mathrm{E}\left[e^2_{a_i}\right]\right)^3 + 45\sigma_v^2\left(\mathrm{E}\left[e^2_{a_i}\right]\right)^2 + 45\xi_v^4\mathrm{E}\left[e^2_{a_i}\right] + 15\xi_v^6. \end{aligned} \tag{66}$$

The expression for $\mathcal{Z}_2$ is related to the desired term $\mathrm{E}[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}^2[e_i]]$ through the equality

$$\begin{aligned} \mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\mathrm{g}^2[e_i]\right] &= \mathcal{Z}_2\mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\right] \\ &= \mathcal{Z}_2\mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2_\Sigma\right]. \end{aligned} \tag{67}$$

Since

$$\begin{aligned} \mathrm{E}\left[\|\mathbf{u}_i\|^2_{\mathrm{H}\Sigma\mathrm{H}}\right] &= \mathrm{E}\left[\mathbf{u}_i\mathrm{H}[\mathbf{u}_i]\Sigma\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}\right] \\ &= \mathrm{E}\left[\mathrm{sign}[\mathbf{u}_i]\Sigma\,\mathrm{sign}\,[\mathbf{u}_i]^{\mathrm{T}}\right] \\ &= \mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2_\Sigma\right]. \end{aligned} \tag{68}$$

Substituting (63) and (67) into (55), we get

$$\begin{aligned} \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_\Sigma\right] = {} & \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_\Sigma\right] + \mu^2\mathcal{Z}_2\mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2_\Sigma\right] \\ & - 2\mu\mathcal{Z}_1\mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}e_{a_i}\right]. \end{aligned} \tag{69}$$

Independence Assumption. If we assume that the regressor sequence $\{\mathbf{u}_i\}$ is i.i.d. then

$$\begin{aligned} \mathrm{E}\left[e^{\mathrm{H}\Sigma}_{a_i}e_{a_i}\right] &= \mathrm{E}\left[\tilde{\mathbf{w}}^{\mathrm{T}}_{i-1}\Sigma\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i\tilde{\mathbf{w}}_{i-1}\right] \\ &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma\mathrm{H}\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i}\right]. \end{aligned} \tag{70}$$

In this way, the terms $\{\mathrm{E}[e_{a_i}^{H\Sigma}e_{a_i}], \mathcal{Z}_1, \mathcal{Z}_2\}$ become all functions of $\tilde{\mathbf{w}}_{i-1}$. Therefore, (69) becomes

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|_\Sigma^2\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|_\Sigma^2\right] + \mu^2\mathcal{Z}_2\mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|_\Sigma^2\right] \\ &\quad - 2\mu\mathcal{Z}_1\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma\mathrm{H}\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i}\right] \\ &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|_\Sigma^2\right] + \mu^2\mathcal{Z}_2\mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|_\Sigma^2\right] \\ &\quad - 2\mu\mathcal{Z}_1\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}\mathbf{u}_i}\right] \\ &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|_\Sigma^2\right] + \mu^2\mathcal{Z}_2\mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|_\Sigma^2\right] \\ &\quad - \sqrt{\frac{8}{\pi\sigma_u^2}}\mu\mathcal{Z}_1\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\Sigma\mathrm{R}}\right].\end{aligned} \tag{71}$$

We thus find that studying the transient behavior of the SRLMF algorithm in effect has reduced to evaluating the functions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ and studying the resulting variance relation (71). Let us now illustrate the application of the above results for white and correlated input data.

White Input Data. For white input data $\mathbf{R}$ is diagonal, say $\mathbf{R} = \sigma_u^2\mathrm{I}$. Therefore, if we select $\Sigma = \mathrm{I}$, the variance relation (71) becomes

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + \mu^2\mathcal{Z}_2\mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2\right] \\ &\quad - \sqrt{\frac{8\sigma_u^2}{\pi}}\mu\mathcal{Z}_1\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right].\end{aligned} \tag{72}$$

Now since

$$\begin{aligned}e_{a_i}^2 &= \tilde{\mathbf{w}}_{i-1}^{\mathrm{T}}\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i\tilde{\mathbf{w}}_{i-1} \\ &= \|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i}.\end{aligned} \tag{73}$$

Substituting (73) into (66), we get

$$\begin{aligned}\mathcal{Z}_2 &= 15\left(\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i}\right]\right)^3 + 45\sigma_v^2\left(\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i}\right]\right)^2 \\ &\quad + 45\xi_v^4\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i}\right] + 15\xi_v^6 \\ &= 15\left(\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathrm{R}}\right]\right)^3 + 45\sigma_v^2\left(\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathrm{R}}\right]\right)^2 \\ &\quad + 45\xi_v^4\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathrm{R}}\right] + 15\xi_v^6 \\ &= 15\left(\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right]\right)^3 + 45\sigma_v^2\left(\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right]\right)^2 \\ &\quad + 45\xi_v^4\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + 15\xi_v^6.\end{aligned} \tag{74}$$

Similarly by substituting (73) into (62), we get

$$\mathcal{Z}_1 = 3\left(\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + \sigma_v^2\right). \tag{75}$$

Substituting (74) and (75) into (72), we get

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + \mu^2\Big[15\left(\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right]\right)^3 \\ &\quad + 45\sigma_v^2\left(\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right]\right)^2 \\ &\quad + 45\xi_v^4\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + 15\xi_v^6\Big] \\ &\quad \times \mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2\right] - 3\sqrt{\frac{8\sigma_u^2}{\pi}}\mu \\ &\quad \times \left(\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + \sigma_v^2\right)\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right].\end{aligned} \tag{76}$$

Since $\mathrm{E}[\|\mathrm{sign}[\mathbf{u}_i]\|^2] = M$, the recursion in (76) becomes

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + 15\mu^2M\sigma_u^6\left(\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right]\right)^3 \\ &\quad + 45\mu^2M\sigma_v^2\sigma_u^4\left(\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right]\right)^2 \\ &\quad + 45\mu^2M\xi_v^4\sigma_u^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + 15\mu^2M\xi_v^6 \\ &\quad - 6\sqrt{\frac{2\sigma_u^2}{\pi}}\mu\sigma_u^2\left(\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right]\right)^2 \\ &\quad - 6\sqrt{\frac{2\sigma_u^2}{\pi}}\mu\sigma_v^2\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] \\ &= f\,\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + 15\mu^2M\xi_v^6,\end{aligned} \tag{77}$$

where

$$\begin{aligned}f &= 1 + 3\mu\left(15\mu M\sigma_u^2\xi_v^4 - 2\sqrt{\frac{2\sigma_u^2}{\pi}}\sigma_v^2\right) \\ &\quad + 3\mu\sigma_u^2\left(15\mu M\sigma_u^2\sigma_v^2 - 2\sqrt{\frac{2\sigma_u^2}{\pi}}\right)\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] \\ &\quad + 15\mu^2M\sigma_u^6\left(\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right]\right)^2.\end{aligned} \tag{78}$$

We see that the transient behavior of the SRLMF algorithm is described by a nonlinear recursion in $\mathrm{E}[\|\tilde{\mathbf{w}}_i\|^2]$ due to the presence of the factor $\mathrm{E}[\|\tilde{\mathbf{w}}_{i-1}\|^2]$ inside f.

Correlated Input Data. For uncorrelated data, the variance relation (72) shows that only unweighted norms of $\tilde{\mathbf{w}}_i$ and $\tilde{\mathbf{w}}_{i-1}$ appear on both sides of the equation. However, for correlated data, different weighing matrices will appear on both sides of (72).

If $\Sigma = \mathrm{I}$ in (71), we get

$$\begin{aligned}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2\right] + \mu^2\mathcal{Z}_2\mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2\right] \\ &\quad - \sqrt{\frac{8}{\pi\sigma_u^2}}\mu\mathcal{Z}_1\mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathrm{R}}\right].\end{aligned} \tag{79}$$

If $\Sigma = \mathbf{R}$ in (71), we get

$$\begin{aligned} \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}}\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{R}}\right] + \mu^2 \mathcal{Z}_2 \mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2_{\mathbf{R}}\right] \\ &\quad - \sqrt{\frac{8}{\pi\sigma_u^2}}\mu \mathcal{Z}_1 \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{R}^2}\right]. \end{aligned} \tag{80}$$

Similarly if $\Sigma = \mathbf{R}^{M-1}$ in (71), we get

$$\begin{aligned} \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}^{M-1}}\right] &= \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{R}^{M-1}}\right] + \mu^2 \mathcal{Z}_2 \mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2_{\mathbf{R}^{M-1}}\right] \\ &\quad - \sqrt{\frac{8}{\pi\sigma_u^2}}\mu \mathcal{Z}_1 \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{R}^{M}}\right]. \end{aligned} \tag{81}$$

The term $\mathrm{E}[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}^M}]$ can be inferred from the prior weighting factors

$$\left\{\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2\right], \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}}\right], \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}^2}\right], \ldots, \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}^{M-1}}\right]\right\}, \tag{82}$$

by expressing $\mathbf{R}^M$ as a linear combination of its lower-order powers using the Cayley-Hamilton theorem. Thus let $p(x) = \det(x\mathrm{I} - \mathbf{R})$ denote the characteristic polynomial of $\mathbf{R}$, say

$$p(x) = x^M + p_{M-1}x^{M-1} + p_{M-2}x^{M-2} + \cdots + p_1 x + p_0. \tag{83}$$

Then we know that [18]:

$$\mathbf{R}^M = -p_{M-1}\mathbf{R}^{M-1} - p_{M-2}\mathbf{R}^{M-2} - \cdots - p_1\mathbf{R} - p_0\mathrm{I}. \tag{84}$$

Using this fact, we have

$$\begin{aligned} \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}^M}\right] &= -p_0 \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2\right] - p_1 \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}}\right] \\ &\quad - \cdots \\ &\quad - p_{M-1}\mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}^{M-1}}\right]. \end{aligned} \tag{85}$$

We can collect the above results into a compact vector notation by writing (79)–(81) as

$$\mathcal{W}_i = \mathcal{F}\mathcal{W}_{i-1} + \mu^2 \mathcal{Z}_2 \mathcal{Y}, \tag{86}$$

where the $M \times 1$ vectors $\{\mathcal{W}_i, \mathcal{Y}\}$ are given by

$$\mathcal{W}_i = \begin{bmatrix} \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2\right] \\ \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}}\right] \\ \vdots \\ \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}^{M-1}}\right] \end{bmatrix}, \qquad \mathcal{Y} = \begin{bmatrix} \mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2\right] \\ \mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2_{\mathbf{R}}\right] \\ \vdots \\ \mathrm{E}\left[\|\mathrm{sign}[\mathbf{u}_i]\|^2_{\mathbf{R}^{M-1}}\right] \end{bmatrix}, \tag{87}$$

and the $M \times M$ coefficient matrix $\mathcal{F}$ is given by

$$\mathcal{F} = \begin{bmatrix} 1 & -\sqrt{\frac{8}{\pi\sigma_u^2}}\mu\mathcal{Z}_1 & & & \\ 0 & 1 & -\sqrt{\frac{8}{\pi\sigma_u^2}}\mu\mathcal{Z}_1 & & \\ 0 & 0 & 1 & -\sqrt{\frac{8}{\pi\sigma_u^2}}\mu\mathcal{Z}_1 & \\ \vdots & & & & \\ 0 & 0 & & 1 & -\sqrt{\frac{8}{\pi\sigma_u^2}}\mu\mathcal{Z}_1 \\ \sqrt{\frac{8}{\pi\sigma_u^2}}\mu p_0\mathcal{Z}_1 & \sqrt{\frac{8}{\pi\sigma_u^2}}\mu p_1\mathcal{Z}_1 & \cdots & \sqrt{\frac{8}{\pi\sigma_u^2}}\mu p_{M-2}\mathcal{Z}_1 & 1 + \sqrt{\frac{8}{\pi\sigma_u^2}}\mu p_{M-1}\mathcal{Z}_1 \end{bmatrix}. \tag{88}$$

As can be seen from (86), the transient behavior of the SRLMF algorithm is described by an M-dimensional state-space recursion as opposed to one-dimensional in the white input case (72).

We know that, the mean-square error is defined as

$$\mathrm{MSE} \triangleq \lim_{i\to\infty} \mathrm{E}\left[|e_i|^2\right], \tag{89}$$

and the excess mean-square error is defined as

$$\mathrm{EMSE} \triangleq \lim_{i\to\infty} \mathrm{E}\left[|e_{a_i}|^2\right], \tag{90}$$

where

$$\mathrm{E}\left[|e_{a_i}|^2\right] = \mathrm{E}\left[\|\tilde{\mathbf{w}}_{i-1}\|^2_{\mathbf{R}}\right]. \tag{91}$$

Table 1: Computational load per iteration for LMF and SRLMF algorithms when data is real.

Algorithm	+	×	Sign
LMF	$2M$	$2M+3$	
SRLMF	$2M$	$2M+2$	1

Table 2: Computational load per iteration for LMF and SRLMF algorithms when data is complex.

Algorithm	+	×	Sign
LMF	$8M+1$	$8M+5$	
SRLMF	$6M+1$	$6M+3$	2

The evolution of $\mathrm{E}[|e_{a_i}|^2]$ is described by the second entry of the state vector $\mathcal{W}_i$ in (86). The resulting learning curve of the filter is $\mathrm{E}[|e_i|^2] = \sigma_v^2 + \mathrm{E}[|e_{a_i}|^2]$.

We know that the mean-square deviation is defined as

$$\mathrm{MSD} \triangleq \lim_{i\to\infty} \mathrm{E}\left[\|\tilde{\mathbf{w}}_i\|^2\right]. \tag{92}$$

The evolution of $\mathrm{E}[\|\tilde{\mathbf{w}}_i\|^2]$ is described by the first entry of the state vector $\mathcal{W}_i$ in (86).

7. Computational Load

Finally, the computational complexity of the LMF and SRLMF algorithms is discussed in this section. Tables 1 and 2 detail the estimated computational load per iteration for LMF and SRLMF algorithms, respectively, for real- and complex-valued data in terms of the number of real additions (+), real multiplications (×), and comparisons with zero (or sign evaluations). We know that one complex multiplication requires four real multiplications and two real additions, while one complex addition requires two real additions.

As can be seen from these two tables, the computational complexity of the SRLMF algorithm becomes more interesting when the data is complex-valued. The case of fading channels in mobile communications is a good example where this scenario can bring drastic improvement in complexity of the SRLMF algorithm over the LMF algorithm.

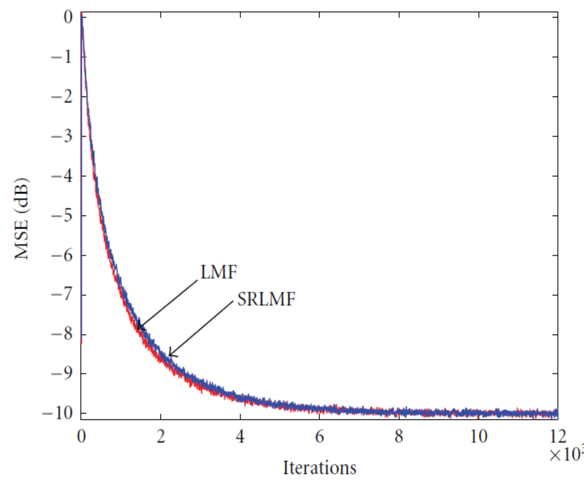

Figure 1: Comparison of the MSE learning curves of LMF and SRLMF algorithms in a uniform noise environment with SNR = 10 dB.

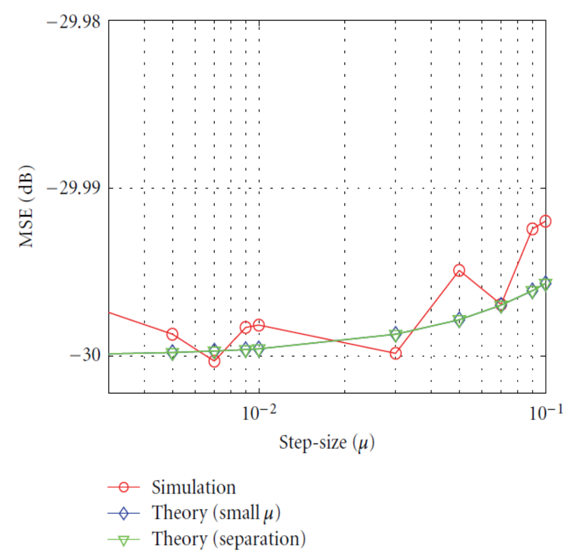

Figure 2: Theoretical and simulated MSE learning curves of the SRLMF algorithm using white Gaussian regressors with shift structure with SNR = 30 dB.

8. Simulation Results

First, the performance analysis of the LMF and the SRLMF algorithms is investigated in an unknown system identification setup with $\mathbf{w}^o = [0.227\quad 0.460\quad 0.688\quad 0.460\quad 0.227]^T$ as far as convergence, steady-state and transient behaviors are concerned. Figure 1 depicts the convergence behavior of the two algorithms for a signal to noise ratio (SNR) of 10 dB in a uniform environment. This figure shows almost identical performance for the two algorithms; no deterioration has occurred to the SRLMF algorithm.

Second, in order to validate the theoretical findings, extensive simulations are carried out for different scenarios. While Figures 2–4 are for the case of the steady-state EMSE of the SRLMF algorithm in a stationary environment, Figure 5 is for the case of the tracking EMSE in a nonstationary environment. In all of these figures the MSE is plotted versus the step-size μ with a SNR = 30 dB.

In the case of Figure 2, the regressors, with shift structure, are generated by feeding a unit-variance white process into a tapped delay line. However, in Figure 3, the regressors, with shift structure, are generated by passing correlated data into a tapped delay line. Here, the correlated data are obtained by passing a unit-variance i.i.d. Gaussian data through a first-order autoregressive model with transfer function $\sqrt{1-a^2}/(1-az^{-1})$ and $a = 0.8$. To further test the validity of the results, Gaussian regressors with an eigenvalue spread of five without a shift structure are used, this is depicted in Figure 4. As it can be seen from these figures, the simulation results match very well the theoretical results ((19) and (20)).

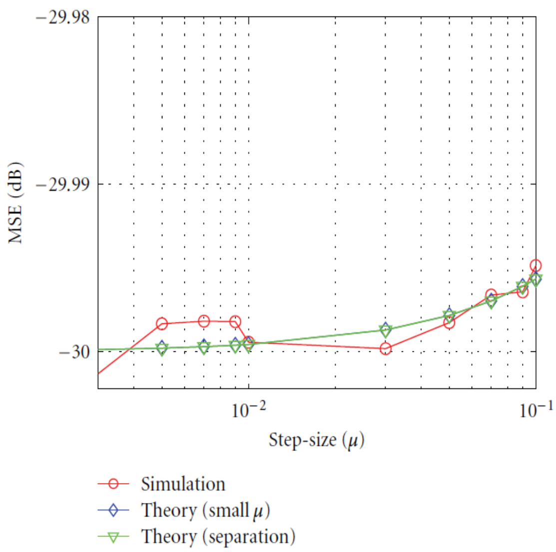

FIGURE 3: Theoretical and simulated MSE learning curves of the SRLMF algorithm using correlated Gaussian regressors with shift structure with SNR = 30 dB.

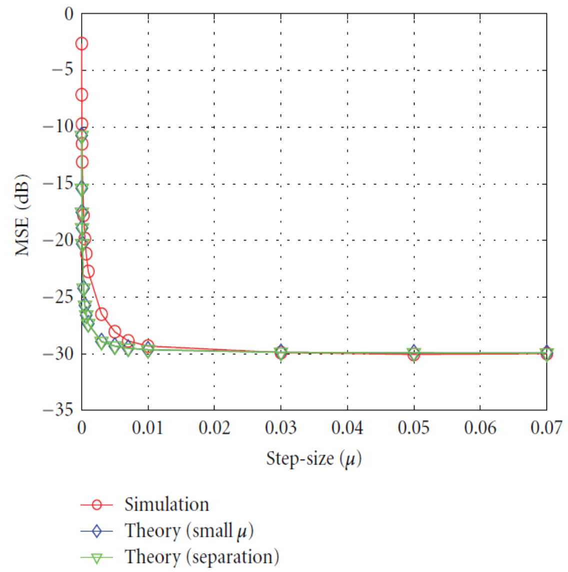

FIGURE 5: Theoretical and simulated MSE learning curves of the SRLMF algorithm for a random-walk channel with SNR = 30 dB.

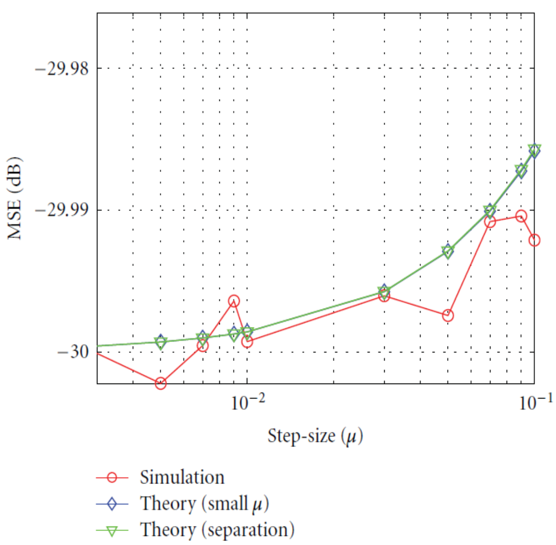

FIGURE 4: Theoretical and simulated MSE learning curves of the SRLMF algorithm using Gaussian regressors with an eigenvalue spread = 5 without shift structure with SNR = 30 dB.

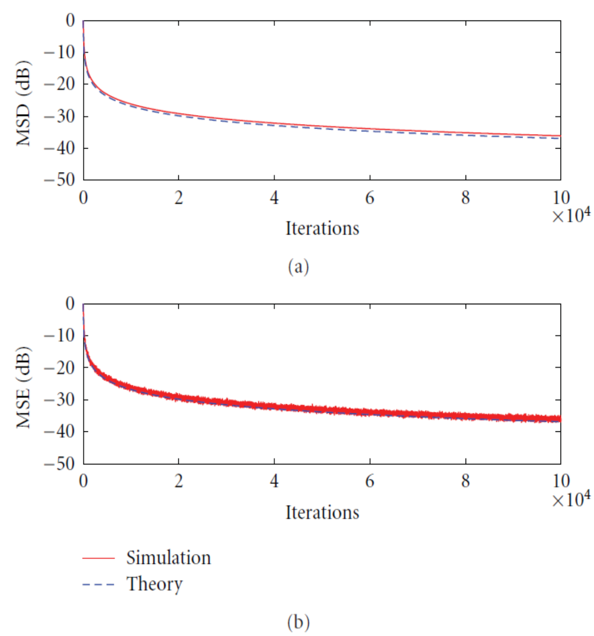

FIGURE 6: Theoretical and simulated MSD (a) and MSE (b) learning curves of the SRLMF algorithm using white Gaussian regressors with SNR = 50 dB.

Third, to further validate the theoretical results in a tracking scenario, the results of Figure 5 depicts this behavior. Here, the random-walk channel behaves according to

$$\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i, \tag{93}$$

where $\mathbf{q}_i$ is a Gaussian sequence with zero mean and variance $\sigma_q^2 = 10^{-9}$ and $\mathbf{w}_{-1}^o = \mathbf{w}^o$. As observed from Figure 5, the simulation results corroborate closely the theoretical results ((34) and (36)).

Finally, we examine the transient behavior of the SRLMF algorithm for the case of Gaussian data. Let us consider a real-valued regression sequence $\{\mathbf{u}_i\}$ with covariance matrix $\mathbf{R}$ whose eigenvalue spread we set at $\rho = 5$. Let the SNR be 50 dB and the step-size is fixed at $\mu = 0.01$.

The results in Figures 6 and 7 show the theoretical and simulated MSD and MSE learning curves of the SRLMF algorithm using white Gaussian regressors and Gaussian regressors with an eigenvalue spread equal to 5. The theoretical values are obtained by using the expression (86). As can be seen here, There is an excellent match between the theoretical and simulated results.

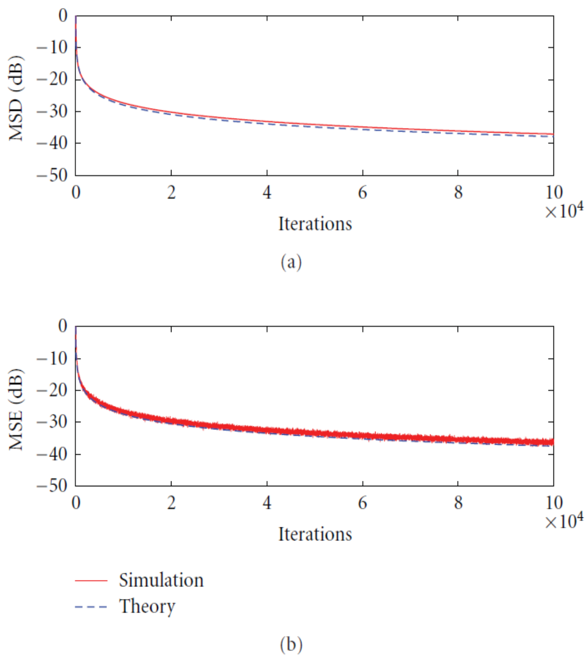

FIGURE 7: Theoretical and simulated MSD (a) and MSE (b) learning curves of the SRLMF algorithm using Gaussian regressors with an eigenvalue spread = 5, SNR = 50 dB.

9. Conclusions

A new adaptive algorithm, called the SRLMF algorithm, has been presented in this work. Expressions are derived for the steady-state EMSE in a stationary environment. A condition for the mean convergence is also found, and it turns out that the convergence of the SRLMF algorithm strongly depends on the choice of initial conditions. Also, expressions are obtained for the tracking EMSE in a nonstationary environment. An optimum value of the step-size μ is also evaluated. Moreover, an extension of the weighted variance relation is provided in order to derive expressions for the mean-square error (MSE) and the mean-square deviation (MSD) of the proposed algorithm during the transient phase. Monte Carlo simulations have shown that there is a good agreement between the theoretical and simulated results. The simulation results indicate that both the SRLMF algorithm and the LMF algorithm converge at the same rate resulting in no performance loss. The analysis developed in this paper is believed to make practical contributions to the design of adaptive filters using the SRLMF algorithm instead of the LMF algorithm in pursuit of the reduction in computational cost and complexity whilst still maintaining good performance.

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Analysis of the Complex Sign Regressor Least Mean Fourth Adaptive Algorithm

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***Abstract*—In this paper, expressions are derived for the steady-state and tracking excess-mean-square error (EMSE) of the complex sign regressor least mean fourth (SRLMF) adaptive algorithm. In addition, an expression for optimum step-size is also derived. Finally, it is shown that the theoretical results are consistent with the simulation results.**

I. Introduction

The sign regressor least mean fourth (SRLMF) adaptive algorithm is based on clipping of the input data. A thorough analysis of the SRLMF algorithm for the case of real-valued data can be found in [1]. Furthermore, it was shown that the SRLMF algorithm and the least mean fourth (LMF) algorithm [2] converge at the same rate for real-valued data resulting in no performance loss.

The present paper extends the mean-square analysis and the tracking analysis of the SRLMF algorithm for the case of complex-valued data. In the process of this evaluation, we distinguished between real- and complex-valued data as the definition of the sign function is different in both cases. The framework used here relies on energy conservation arguments [3]. It is shown that the analytical results of the paper are in a good agreement with the simulation results. Moreover, the results show that the complex SRLMF algorithm has a performance loss in terms of convergence behavior when compared with the complex LMF algorithm.

II. The Complex SRLMF Algorithm

The complex SRLMF algorithm is based on clipping of the regressor $\mathbf{u}_i$. The update equation of the algorithm is given by

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \ \mathrm{csgn}[\mathbf{u}_i]^* e_i |e_i|^2, \qquad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ is the updated weight vector, μ is the step-size, e_i is the estimation error signal, and

$$\mathrm{csgn}[x] = \begin{cases} -1, & \text{if } \Re[x] < 0 \text{ or} \\ & (\Re[x] = 0 \text{ and } \Im[x] < 0), \\ 0, & \text{if } \Re[x] = \Im[x] = 0, \\ 1, & \text{if } \Re[x] > 0 \text{ or} \\ & (\Re[x] = 0 \text{ and } \Im[x] > 0). \end{cases} \tag{2}$$

III. Mean-Square Analysis of the Complex SRLMF Algorithm

Let us assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the stationary data model [3]:

- A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that $d_i = \mathbf{u}_i \mathbf{w}^o + v_i$.
- A.2 The noise sequence v_i is independent and identically distributed (i.i.d.) circular with variance $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ and is independent of $\mathbf{u}_j$ for all i, j.
- A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.
- A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}\left[\mathbf{u}_i^* \mathbf{u}_i\right] > \mathbf{0}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [3]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right] = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]\right], \quad \text{as } i \to \infty, \tag{3}$$

where

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \mathrm{E}[\mathrm{Re}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i] \mathbf{u}_i^*]], \tag{4}$$

$$e_i = e_{a_i} + v_i, \tag{5}$$

and $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error. Then $\mathrm{g}[e_i]$ becomes

$$\begin{aligned} \mathrm{g}[e_i] \;\; = \;\; & e_{a_i}|e_{a_i}|^2 + e_{a_i}|v_i|^2 + e_{a_i}[e_{a_i}^* v_i + e_{a_i} v_i^*] + v_i |e_{a_i}|^2 \\ & + v_i |v_i|^2 + v_i [e_{a_i}^* v_i + e_{a_i} v_i^*]. \end{aligned} \tag{6}$$

By using the fact that e_{a_i} and v_i are independent, we reach at the following expression for the term $\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]$:

$$\begin{aligned} \mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right] \;\; = \;\; & \mathrm{E}\left[|e_{a_i}|^2 |v_i|^2\right] + \mathrm{E}\left[|e_{a_i}|^2 [e_{a_i}^* v_i + e_{a_i} v_i^*]\right] \\ & + \mathrm{E}\left[e_{a_i}^* v_i [e_{a_i}^* v_i + e_{a_i} v_i^*]\right]. \end{aligned} \tag{7}$$

Ignoring third and higher-order terms of e_{a_i} and since the noise sequence v_i is assumed to be circular i.e., $\mathrm{E}[v_i^2] = 0$, then (7) becomes

$$\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right] \approx 2\sigma_v^2 \mathrm{E}[|e_{a_i}|^2]. \tag{8}$$

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To evaluate the term $\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|\mathrm{g}[e_i]|^2\right]$, we start by noting that

$$\begin{aligned}|\mathrm{g}[e_i]|^2 &= |e_{a_i}|^6+|v_i|^6+3|e_{a_i}|^4[e^*_{a_i}v_i+e_{a_i}v^*_i]\\ &+3|v_i|^4[e^*_{a_i}v_i+e_{a_i}v^*_i]+|e_{a_i}|^2e^*_{a_i}v_i[3e^*_{a_i}v_i\\ &+2e_{a_i}v^*_i]+|e_{a_i}|^2e_{a_i}v^*_i[3e_{a_i}v^*_i+2e^*_{a_i}v_i]\\ &+5|v_i|^2|e_{a_i}|^4+5|e_{a_i}|^2|v_i|^4+|v_i|^2e^*_{a_i}v_i[3e^*_{a_i}v_i\\ &+2e_{a_i}v^*_i]+|v_i|^2e_{a_i}v^*_i[3e_{a_i}v^*_i+2e^*_{a_i}v_i]\\ &+9|v_i|^2|e_{a_i}|^2[e^*_{a_i}v_i+e_{a_i}v^*_i]+e^{*3}_{a_i}v^3_i+e^3_{a_i}v^{*3}_i.\end{aligned} \tag{9}$$

If we multiply $|\mathrm{g}[e_i]|^2$ by $||\mathbf{u}_i||^2_{\mathrm{H}}$ from the left, use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, and again ignoring third and higher-order terms of e_{a_i}, we get

$$\begin{aligned}\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|\mathrm{g}[e_i]|^2\right] &\approx \xi^6_v\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}]+9\xi^4_v\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}|e_{a_i}|^2]\\ &+3\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|v_i|^4[e^*_{a_i}v_i+e_{a_i}v^*_i]\right],\end{aligned} \tag{10}$$

where $\xi^4_v=\mathrm{E}[|v_i|^4]$, $\xi^6_v=\mathrm{E}[|v_i|^6]$ denote the forth and sixth-order moments of v_i, respectively.
From Price's theorem [4] we have

$$\mathrm{E}\left[\mathrm{Re}[x^*\mathrm{csgn}(y)]\right]=\sqrt{\frac{2}{\pi}}\frac{\sqrt{2}}{\sigma_y}\mathrm{E}\left[\mathrm{Re}[x^*y]\right], \tag{11}$$

where x and y denote two complex-valued jointly-Gaussian random variables. Therefore,

$$\begin{aligned}\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}] &= \mathrm{E}[\mathrm{Re}[\mathbf{u}_i\mathrm{H}[\mathbf{u}_i]\mathbf{u}^*_i]],\\ &= \mathrm{E}\left[\mathrm{Re}[\mathbf{u}_i\mathrm{csgn}[\mathbf{u}_i]^*]\right],\\ &= \frac{4\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma^2_u}}.\end{aligned} \tag{12}$$

Substituting (12) into (10) we get

$$\begin{aligned}\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|\mathrm{g}[e_i]|^2\right] &\approx \xi^6_v\left[\frac{4\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma^2_u}}\right]+9\xi^4_v\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}|e_{a_i}|^2]\\ &+3\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|v_i|^4[e^*_{a_i}v_i+e_{a_i}v^*_i]\right].\end{aligned} \tag{13}$$

Substituting (8) and (13) into (3) we get

$$\begin{aligned}4\sigma^2_v\mathrm{E}[|e_{a_i}|^2] &= \mu\xi^6_v\left[\frac{4\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma^2_u}}\right]+9\mu\xi^4_v\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}|e_{a_i}|^2]\\ &+3\mu\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|v_i|^4[e^*_{a_i}v_i+e_{a_i}v^*_i]\right].\end{aligned} \tag{14}$$

In order to simplify (14) and arrive at an expression for the steady-state excess-mean-square error (EMSE) $\zeta=\mathrm{E}[|e_{a_i}|^2]$ of the complex SRLMF algorithm, let us consider the following two cases:

1. Sufficiently small step-sizes

Small step-sizes lead to small values of $\mathrm{E}[|e_{a_i}|^2]$ and e_{a_i} in steady-state. Therefore, in this case, the last two terms in (14) can be ignored, the steady-state EMSE is given by

$$\zeta=\frac{\mu\xi^6_v\mathrm{Tr}(\mathbf{R})}{\sigma^2_v\sqrt{\pi\sigma^2_u}}. \tag{15}$$

2. Separation principle

For larger step-sizes, we use the separation assumption, namely, that at steady-state, $||\mathbf{u}_i||^2_{\mathrm{H}}$ is independent of e_{a_i}. In this case, the last term in (14) will be zero since e_{a_i} is zero mean, the steady-state EMSE can be shown to be

$$\zeta=\frac{\mu\xi^6_v\left[\frac{4}{\sqrt{\pi\sigma^2_u}}\right]\mathrm{Tr}(\mathbf{R})}{\left(4\sigma^2_v-9\mu\xi^4_v\left[\frac{4}{\sqrt{\pi\sigma^2_u}}\right]\mathrm{Tr}(\mathbf{R})\right)}. \tag{16}$$

IV. Tracking Analysis of the Complex SRLMF Algorithm

Here, let us assume that the data $\{d_i,\mathbf{u}_i\}$ satisfy the following assumptions of the nonstationary data model [3]:

- A.5 There exists a vector $\mathbf{w}^o_i$ such that $d_i=\mathbf{u}_i\mathbf{w}^o_i+v_i$.
- A.6 The weight vector varies according to the random-walk model $\mathbf{w}^o_i=\mathbf{w}^o_{i-1}+\mathbf{q}_i$, and the sequence $\mathbf{q}_i$ is i.i.d. with covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j,\mathbf{u}_j\}$ for all i,j.
- A.7 The initial conditions $\{\mathbf{w}_{-1},\mathbf{w}^o_{-1}\}$ are independent of the zero mean random variables $\{d_i,\mathbf{u}_i,v_i,\mathbf{q}_i\}$.

In this case, the following variance relation holds [3]:

$$\begin{aligned}\mu\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|\mathrm{g}[e_i]|^2\right]+\mu^{-1}\mathrm{Tr}(\mathbf{Q}) &= 2\mathrm{Re}\left[\mathrm{E}\left[e^*_{a_i}\mathrm{g}[e_i]\right]\right],\\ &\quad\text{as } i\to\infty.\end{aligned} \tag{17}$$

Tracking results can be obtained by inspection from the mean-square results as there are only minor differences. Therefore, by substituting (8) and (13) into (17) we get

$$\begin{aligned}4\sigma^2_v\mathrm{E}[|e_{a_i}|^2] &= \mu^{-1}\mathrm{Tr}(\mathbf{Q})+\mu\xi^6_v\left[\frac{4\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma^2_u}}\right]\\ &+9\mu\xi^4_v\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}|e_{a_i}|^2]\\ &+3\mu\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|v_i|^4[e^*_{a_i}v_i+e_{a_i}v^*_i]\right]\end{aligned} \tag{18}$$

We again consider two cases for the evaluation of the tracking EMSE ζ of the complex SRLMF algorithm:

1. Sufficiently small step-sizes

Also, here, in this case we get

$$\zeta=\frac{\mu^{-1}\mathrm{Tr}(\mathbf{Q})+\mu\xi^6_v\left[\frac{4}{\sqrt{\pi\sigma^2_u}}\right]\mathrm{Tr}(\mathbf{R})}{4\sigma^2_v}. \tag{19}$$

An optimum value of the step-size of the complex SRLMF algorithm is obtained by minimizing (19) with respect to μ. Therefore,

$$\mu_{\mathrm{opt}}=\sqrt{\frac{\sqrt{\pi\sigma^2_u}\mathrm{Tr}(\mathbf{Q})}{4\xi^6_v\mathrm{Tr}(\mathbf{R})}}. \tag{20}$$

2. Separation principle

Similarly here as it was done for the derivation of (16), we obtain the following:

$$\zeta=\frac{\mu^{-1}\mathrm{Tr}(\mathbf{Q})+\mu\xi^6_v\left[\frac{4}{\sqrt{\pi\sigma^2_u}}\right]\mathrm{Tr}(\mathbf{R})}{\left(4\sigma^2_v-9\mu\xi^4_v\left[\frac{4}{\sqrt{\pi\sigma^2_u}}\right]\mathrm{Tr}(\mathbf{R})\right)}, \tag{21}$$

and eventually the optimum step-size of the complex SRLMF algorithm is given by

$$\mu_{\text{opt}} = \sqrt{\mathrm{Tr}(\mathbf{Q})\left[\frac{81(\xi_v^4)^2\mathrm{Tr}(\mathbf{Q})}{16(\sigma_v^2)^2(\xi_v^6)^2} + \frac{\sqrt{\pi\sigma_u^2}}{4\xi_v^6\mathrm{Tr}(\mathbf{R})}\right]} - \frac{9\xi_v^4}{4\sigma_v^2\xi_v^6}\mathrm{Tr}(\mathbf{Q}). \tag{22}$$

V. Simulation Results

First, in order to validate the theoretical findings extensive simulations are carried out for different scenarios. Figure 1 is for the case of the steady-state mean-square error (MSE) of a 5-tap SRLMF filter in a stationary environment and Figure 2 is for the case of the tracking MSE in a nonstationary environment. In both these figures, the MSE is plotted as a function of the step-size μ for a signal to noise ratio (SNR) of 30 dB. Moreover, all the simulations reported in this work use complex-valued signals and white Gaussian regressors with shift structure. As can be seen from Figure 1, the simulation results match very well the theoretical results ((15) and (16)).

Second, to further validate the theoretical results in a tracking scenario, Figure 2 depicts this behavior. Here, the random-walk channel behaves according to

$$\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i, \tag{23}$$

where $\mathbf{q}_i$ is a Gaussian sequence with zero mean and variance $\sigma_q^2 = 10^{-9}$. As observed from Figure 2, the simulation results corroborate closely the theoretical results ((19) and (21)).

Finally, the convergence behavior of the complex LMF and the complex SRLMF algorithms is compared in an uniform noise environment for an SNR of 10 dB, Figure 3 illustrates this behavior. As can be seen from this figure, the complex SRLMF algorithm has a performance loss when compared with the complex LMF algorithm for the same steady-state MSE.

VI. Conclusions

It is interesting to note that the expressions for the steady-state EMSE, tracking EMSE, and optimum step-size of the SRLMF algorithm for real- and complex-valued data are found to be identical except for a scaling factor. It is shown that the simulation results are in a good match with the analytical results. It is also shown that the convergence performance of the complex SRLMF algorithm is inferior when compared with the complex LMF algorithm.

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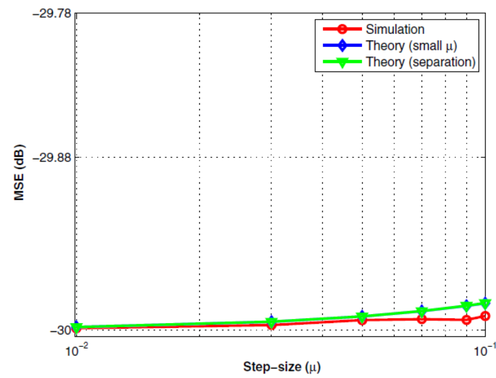

Fig. 1. Theoretical and simulated steady-state MSE of the complex SRLMF algorithm using white Gaussian regressors.

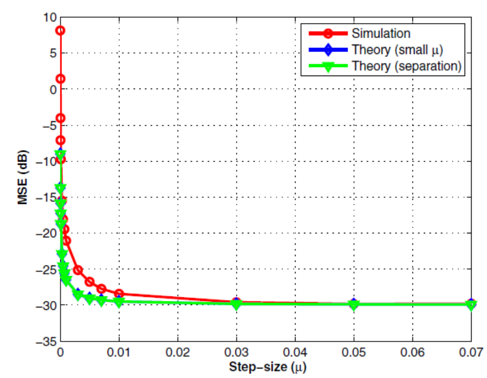

Fig. 2. Theoretical and simulated tracking MSE of the complex SRLMF algorithm using white Gaussian regressors.

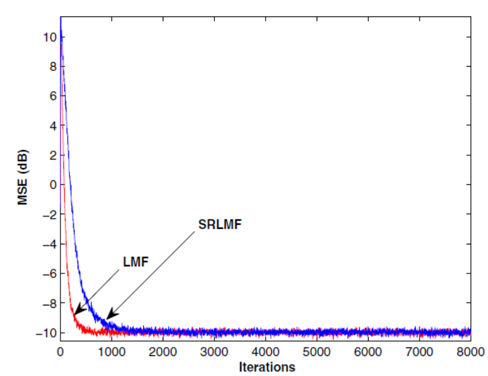

Fig. 3. Comparison of the MSE learning curves of the complex LMF and the complex SRLMF algorithms in a uniform noise environment.

[P3]

ADAPTIVE CHANNEL EQUALIZATION USING THE SIGN REGRESSOR LEAST MEAN FOURTH ALGORITHM

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ABSTRACT

In this paper, the performance analysis of the least mean fourth (LMF) algorithm and the sign regressor least mean fourth (SRLMF) algorithm is investigated in an adaptive channel equalization scenario. The simulation results indicate that both the LMF and the SRLMF algorithms exhibit similar bit error rate (BER) performance. Moreover, the results show that the SRLMF algorithm has a slight performance degradation in terms of convergence behavior when compared with the LMF algorithm.

1. INTRODUCTION

Reduction in complexity of the least mean square (LMS) algorithm has always received attention in the area of adaptive filtering [1]–[3]. This reduction is usually done by clipping either the estimation error or the input data, or both to reduce the number of multiplications necessary at each algorithm iteration. Many complexity reduction algorithms have been reported in the literature and among them is the sign regressor algorithm (SRA), which is based on clipping of the input data [4]–[7]. These algorithms result in a performance loss when compared with the LMS algorithm [4]–[5]. However, significant reduction in computational cost and simplified hardware implementation can justify this poor performance in applications requiring reduced implementation costs [8]–[9].

In this work, the bit error rate (BER) performance and the convergence performance of the least mean fourth (LMF) algorithm [10], and the sign regressor least mean fourth (SRLMF) algorithm [11] in an adaptive channel equalization scenario [12] are compared. Moreover, the convergence analysis, the mean-square analysis, the tracking analysis, and the transient analysis of the SRLMF algorithm are found in [11].

This paper is organized as follows: following the Introduction is Section 2 where the SRLMF algorithm is described. The computational complexity is detailed in Section 3. The adaptive linear equalization and decision-feedback equalization (DFE) scenarios are presented in Sections 4 and 5, respectively. The simulation results are reported in Section 6. Finally, some conclusions are reported in Section 7.

2. THE SRLMF ALGORITHM

The SRLMF algorithm is based on clipping of the regressor $\mathbf{u}_i$, which is a row vector, say

$$\mathbf{u}_i \triangleq [u_i \quad u_{i-1} \quad u_{i-2} \quad \dots \quad u_{i-M+1}], \qquad (1)$$

where M is the filter order. For real-valued data, the SRLMF update recursion is

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\,\text{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \qquad i \geq 0, \qquad (2)$$

where $\mathbf{w}_i$ (column vector) is the updated weight vector at time i, μ is the step-size, e_i is the estimation error signal, and $\text{sign}[\mathbf{u}_i]$ is a row vector with the signs of the entries of $\mathbf{u}_i$ defined as

$$\text{sign}[x] = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases} \qquad (3)$$

For complex-valued data, the update recursion in (2) becomes

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\,\text{csgn}[\mathbf{u}_i]^* e_i^3, \qquad i \geq 0, \qquad (4)$$

where

$$\text{csgn}[x] = \begin{cases} -1, & \text{if } \Re[x] < 0 \text{ or} \\ & (\Re[x] = 0 \text{ and } \Im[x] < 0), \\ 0, & \text{if } \Re[x] = \Im[x] = 0, \\ 1, & \text{if } \Re[x] > 0 \text{ or} \\ & (\Re[x] = 0 \text{ and } \Im[x] > 0). \end{cases} \qquad (5)$$

In this work, for binary phase-shift keying (BPSK) and quadrature phase-shift keying (QPSK) data, the SRLMF update equations ((2) and (4)) are used, respectively.

3. COMPUTATIONAL COMPLEXITY

The computational complexity of the LMF and the SRLMF algorithms is discussed in this section. Tables 1 and 2 detail the estimated computational complexity per iteration for the LMF and the SRLMF algorithms, respectively, for real- and complex-valued data in terms of the number of

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Table 1. Computational complexity per iteration of the LMF and the SRLMF algorithms for real-valued data.

Algorithm	$+$	$\times$		**sign**
LMF	$2M$	$2M+3$		
SRLMF	$2M$	$2M+2$	$(M+2)$	1

Table 2. Computational complexity per iteration of the LMF and the SRLMF algorithms for complex-valued data.

Algorithm	$+$	$\times$		**sign**
LMF	$8M+1$	$8M+5$		
SRLMF	$6M+1$	$6M+3$	$(4M+3)$	2

real additions ($+$), real multiplications ($\times$), and sign evaluations.

When the step-size is selected as $\mu = 2^{-m}$ for some positive integer m, then the SRLMF algorithm can be very efficiently implemented by means of shift registers, and the M multiplications that are needed for a generic μ can be ignored. In this particular case, we can replace the $2M$ and $6M$ figures that appear in Tables 1 and 2, respectively, by M and $4M$ multiplications. Therefore, in the multiplications column, the entries between parenthesis indicate the number of multiplications needed whenever the step-size is chosen as a power of 2^{-1}. This simplification is not possible in the case of LMF.

4. ADAPTIVE LINEAR EQUALIZATION

Consider a finite impulse response (FIR) channel with M taps defined by:

$$C(z) = c_0 + c_1 z^{-1} + \ldots + c_{M-1} z^{-(M-1)}. \quad (6)$$

Data symbols s_i transmitted through the channel in (6) get corrupted by the noise sequence v_i, which is assumed to be independent and identically distributed (i.i.d.) additive white Gaussian noise (AWGN). Also, in our work, data symbols s_i are chosen from a BPSK or a QPSK constellation. Due to the channel memory, each received signal u_i contains contributions not only from s_i but also from prior symbols, since

$$u_i = c_0 s_i + \sum_{k=1}^{M-1} c_k s_{i-k} + v_i. \quad (7)$$

The second term on the right-hand side of (7) is the intersymbol-interference (ISI) [12]–[13]. As the name implies it refers to the interference that is caused by prior symbols. The objective of an equalizer is to combat this ISI and to recover the data symbols s_i from the received signal u_i. In order to achieve this objective, an adaptive linear equalizer employs current and prior measurements u_{i-k}, say for $k = 0, 1, \ldots, L-1$. This is because prior measurements contain information that is correlated with the ISI term in u_i, and therefore they can help in estimating the interference term and removing its effect.

The adaptive linear equalizer structure shown in Figure 1 has basically two modes of operation [12]–[13]: a training mode during which a delayed replica of the input sequence s_i is used as a reference sequence d_i, and a decision-directed mode during which the output of the decision-device $\check{s}_{i-\triangle}$ replaces the reference sequence. The received signal u_i is processed by the FIR equalizer to generate the estimated signal $\hat{s}_{i-\triangle}$, which is later fed into a decision device. The purpose of this decision device is to map each $\hat{s}_{i-\triangle}$ to the closest symbol in the symbol constellation.

5. ADAPTIVE DECISION-FEEDBACK EQUALIZATION

A basic adaptive decision-feedback equalization model is shown in Figure 2, which is better suited for channels with pronounced ISI. Unlike an adaptive linear equalizer, a DFE structure uses the prior symbols s_{i-k} themselves, say for $k = 1, \ldots, M-1$, in order to cancel their effect from u_i rather than relying on the prior measurements. In addition to using an FIR filter in the feedforward path, as in adaptive linear equalization, a DFE structure employs a feedback filter in order to feedback previous decisions and use them to reduce ISI. Here, the estimated signal $\hat{s}_{i-\triangle}$ is obtained by combining the outputs of the feedforward and feedback filters [12]–[13].

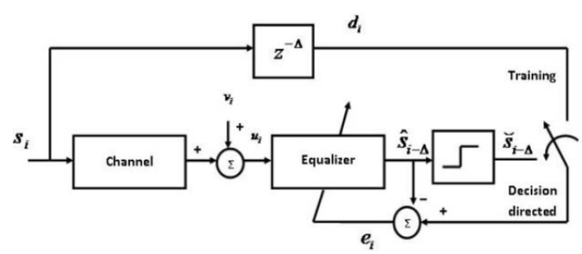

Fig. 1. Adaptive linear equalization model.

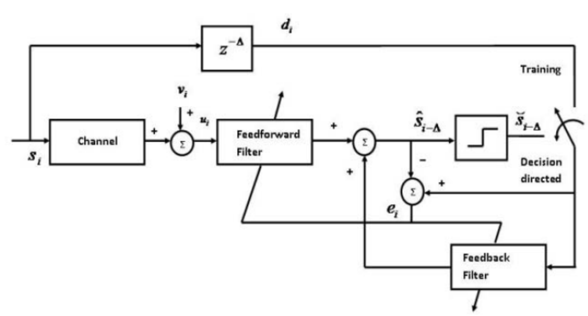

Fig. 2. Adaptive decision feedback-equalization model.

6. SIMULATION RESULTS

First, the performance analysis of the LMF and the SRLMF algorithms is investigated in an adaptive linear equalization setup, respectively, for BPSK and QPSK data. The simulations reported in this work are based on an FIR channel with $M = 4$ taps defined by:

$$C(z) = 0.5 + 1.2z^{-1} + 1.5z^{-2} - z^{-3}. \quad (8)$$

The adaptive filter is trained with 1000 BPSK/QPSK symbols, followed by decision-directed operation during 7000 BPSK/QPSK symbols. Choose delay $\triangle = 7$ and equalizer length $L = 10$. Use SRLMF to train the equalizer with step-size $\mu = 0.01$. The performance of the LMF and the SRLMF algorithms is compared for the same steady-state mean-square error (MSE) or misadjustment.

Finally, the performance analysis of the LMF and the SRLMF algorithms is further investigated in an adaptive DFE setup, respectively, for BPSK and QPSK data. Here, choose $\triangle = 7$, $L = 10$ feedforward taps, and $Q = 2$ feedback taps.

Figure 3 reports the comparison of the BER curves at the input of the LMF linear equalizer and the SRLMF linear equalizer, respectively, for BPSK and QPSK data in an AWGN environment. While Figure 5 reports the same comparison for the case of the LMF DFE and the SRLMF DFE. As can be seen from these figures, for a given constellation, the BER performance of the LMF and the SRLMF algorithms is similar for lower signal to noise ratios (SNR). As the SNR increases, an enhancement in the BER performance of the SRLMF algorithm is obtained over the LMF algorithm. However, the BER performance of the LMF and the SRLMF algorithms is again similar for higher SNR's. Also, one can notice that, for a fixed SNR, the probability of error increases as the order of the constellation increases.

Figure 4 reports the comparison of the MSE learning curves of the LMF linear equalizer and the SRLMF linear equalizer, respectively, for BPSK and QPSK data in a uniform noise environment with SNR = 20 dB. While Figure 6 reports the same comparison for the case of the LMF DFE and the SRLMF DFE. As can be seen from these figures, the convergence performance of the SRLMF algorithm is slightly inferior when compared with the LMF algorithm. Also, one can notice that, the mean square error increases as the order of the constellation increases.

7. CONCLUSIONS

In this work, the performance analysis of the LMF and the SRLMF algorithms in an adaptive channel equalization scenario is compared. The simulation results report similar BER performance for both the algorithms. It is also shown that the convergence performance of the SRLMF algorithm is only slightly inferior when compared with the LMF algorithm.

8. ACKNOWLEDGMENT

The authors acknowledge the support provided by King Fahd University of Petroleum and Minerals to carry out this work.

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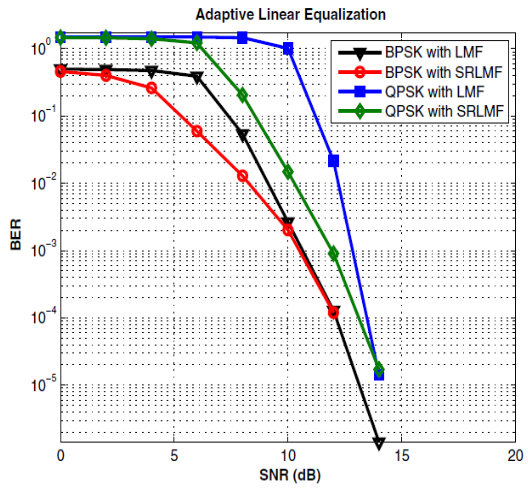

Fig. 3. Comparison of the BER curves at the input of the LMF linear equalizer and the SRLMF linear equalizer, respectively, for BPSK and QPSK data in an AWGN environment.

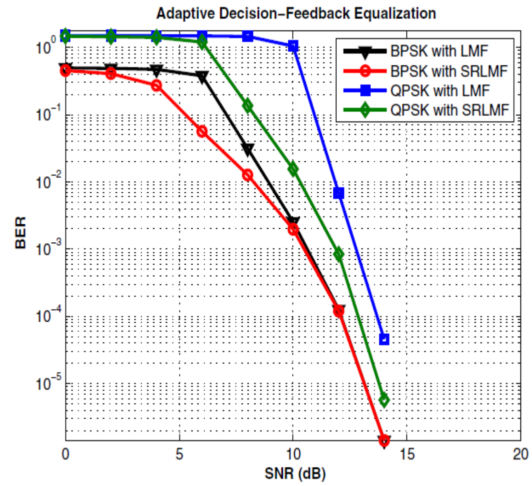

Fig. 5. Comparison of the BER curves at the input of the LMF DFE and the SRLMF DFE, respectively, for BPSK and QPSK data in an AWGN environment.

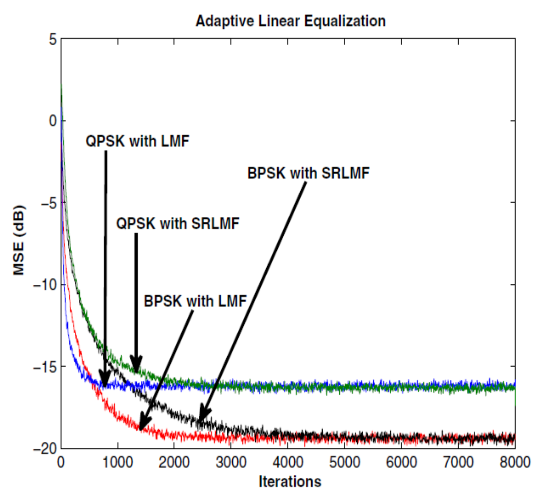

Fig. 4. Comparison of the MSE learning curves of the LMF linear equalizer and the SRLMF linear equalizer, respectively, for BPSK and QPSK data in a uniform noise environment with SNR=20 dB.

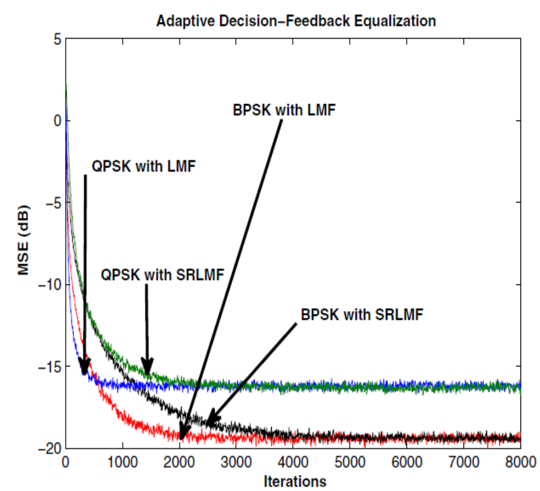

Fig. 6. Comparison of the MSE learning curves of the LMF DFE and the SRLMF DFE, respectively, for BPSK and QPSK data in a uniform noise environment with SNR=20 dB.

3 The SRLMMN Algorithm

3.1 Introduction

The Sign Regressor Least Mean Mixed-Norm (SRLMMN) algorithm is based on the clipping of the input data. The SRLMMN algorithm belongs to the family of the Least Mean Mixed-Norm (LMMN) algorithm. The only difference in the filter weights update equations of these two algorithms is the application of the signum function on the input data of the SRLMMN algorithm. The SRLMMN algorithm is a hybrid version of the Sign Regressor Least Mean Square (SRLMS) and Sign Regressor Least Mean Fourth (SRLMF) algorithms. The SRLMMN algorithm combines the benefits of both the SRLMS and SRLMF algorithms such as improved stability and convergence performance, respectively.

The filter weights update equation of the SRLMMN algorithm for real-valued data is given by (3.1) [34]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i [\delta + (1-\delta) e_i^2], \tag{3.1}$$

where $\mathbf{w}_i$ is the updated filter weight vector at iteration $i \geq 0$, μ is the step-size, $\mathbf{u}_i$ is the regressor vector, $e_i = d_i - y_i$ is the estimation error signal, d_i is the desired signal, y_i is the adaptive filter output, δ is the mixing parameter ranging between $0 \leq \delta \leq 1$, $\mathrm{sign}(.)$ denotes the sign of its argument, and the definition of the signum function for real-valued data is given by (1.1).

The filter weights update equation of the SRLMMN algorithm reduces to the filter weights update equations of the SRLMF and SRLMS algorithms when the mixing parameter δ becomes 0 and 1, respectively.

3.2 Background

The LMMN algorithm is a well-known member of the family of mixed-norm adaptive filtering algorithms and has been analyzed extensively in the open literature. However, there were no efforts made to analyze the performance evaluation of the SRLMMN algorithm until it was proposed, analyzed, and evaluated in [34].

The LMMN algorithm combines the benefits of both the classical Least Mean Square (LMS) and Least Mean Fourth (LMF) algorithms [10]. Some of the studies, which investigated the performance evaluation of the LMMN algorithm are as follows. The convergence, steady-state, and tracking analysis of the LMMN algorithm was studied in [11], [12]. In [58], the LMMN algorithm was introduced for the first time in an adaptive echo canceller wherein it has shown improved performance over the LMS algorithm by offering relatively faster convergence and lower Mean Square Error (MSE).

In [59], an LMMN-based adaptive control technique is employed for the eradication of harmonics and for the extraction of fundamental load component for power quality improvement of grid intertie wind–photovoltaic system. The LMMN-based adaptive control technique offers lesser MSE, thereby resulting in reduced misadjustments and improved convergence compared to the conventional control schemes [59].

The motivation to introduce the sign regressor term in the SRLMMN algorithm is to achieve reduced computational complexity compared to the LMMN algorithm. However, the convergence performance of the SRLMMN algorithm is slower than the SRLMF algorithm but better than the SRLMS algorithm as expected.

3.3 Contributions/Published Manuscripts

The two published papers on the performance evaluation of the SRLMMN [34], [48] algorithm for real-valued data are as follows:

[P4] M. M. U. Faiz and A. Zerguine, "On the convergence, steady-state, and tracking analysis of the SRLMMN algorithm," ***in Proc. of the 23rd European Signal Processing Conf. (EUSIPCO 2015),*** **Nice, France, pp. 2691–2695, Aug.-Sep. 2015, DOI: https://doi.org/10.1109/EUSIPCO.2015.7362873**

A novel adaptive algorithm called the SRLMMN algorithm was proposed, analyzed, and evaluated for the case of real-valued data in [34]. The expressions for the steady-state and tracking MSE of the SRLMMN algorithm were derived and are given by (B.1) and (B.4) in Appendix B, respectively.

In addition, a sufficient condition for the convergence in the mean of the SRLMMN algorithm was also derived and is given by (3.2) [34]:

$$0 < \mu_{\text{SRLMMN}} < \frac{\sqrt{2\pi\sigma_u^2}}{\lambda_{\max}(\delta+3(1-\delta)\sigma_e^2)}, \tag{3.2}$$

where μ_{SRLMMN} is the step-size of the SRLMMN algorithm, $\sigma_u^2 = \mathrm{E}[\mathbf{u}_i{}^2]$ is the regressor variance, $\lambda_{\max}$ is the maximum eigenvalue of the regressor covariance matrix $\mathbf{R}$, and σ_e^2 is the estimation error variance.

We can obtain the expressions for the step-size bounds of the SRLMF and SRLMS algorithms from (3.2) by setting δ equal to 0 and 1, respectively, as shown below:

$$0 < \mu_{\text{SRLMF}} < \frac{\sqrt{2\pi\sigma_u^2}}{3\lambda_{\max}\sigma_e^2}, \tag{3.3}$$

$$0 < \mu_{\mathrm{SRLMS}} < \frac{\sqrt{2\pi\sigma_u^2}}{\lambda_{\max}}. \tag{3.4}$$

Note that the expression for the step-size bound of the SRLMF algorithm in (3.3) is the same as that obtained by the author in [32]. Furthermore, an excellent agreement is observed between the simulation and analytical results.

Finally, a comparison between the convergence performance of the SRLMMN and LMMN algorithms indicates no performance degradation of the SRLMMN algorithm for real-valued data in a uniform noise environment with an SNR of 10 dB.

[P5] M. M. U. Faiz and I. Kale, "Removal of multiple artifacts from ECG signal using cascaded multistage adaptive noise cancellers," ***Array,*** **vol. 14, art. no. 100133, pp. 1–9, July 2022, DOI: https://doi.org/10.1016/j.array.2022.100133**

Although cascaded multistage adaptive noise cancellers have been employed before by other researchers for multiple artifact removal from the ElectroCardioGram (ECG) signal, they all used the same adaptive algorithm in all the cascaded multi-stages for adjusting the adaptive filter weights. In this paper, a cascaded 4-stage adaptive noise canceller is proposed by the author for the removal of four artifacts present in the ECG signal, namely baseline wander, motion artifacts, muscle artifacts, and 60 Hz Power Line Interference (PLI) [48].

The performance of eight adaptive algorithms, namely LMS, LMF, LMMN, SRLMS, Sign-Error Least Mean Square (SLMS), Sign-Sign Least Mean Square (SSLMS), SRLMF, and SRLMMN is investigated in terms of Signal-to-Noise Ratio (SNR) improvement for removing the aforementioned four artifacts from the ECG signal [48].

The shortlisted LMMN, LMF, LMMN, LMF algorithms are employed in the proposed cascaded 4-stage adaptive noise canceller to remove the respective ECG artifacts as mentioned above. The proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF algorithms outperforms those that employ the same algorithm such as the LMS algorithm in all the four stages. One unique and powerful feature of the proposed cascaded 4-stage adaptive noise canceller is that it employs only those adaptive algorithms in the four stages, which are shown to be effective in removing the respective ECG artifacts as mentioned above. Such a scheme has not been investigated before in the open literature [48].

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ON THE CONVERGENCE, STEADY-STATE, AND TRACKING ANALYSIS OF THE SRLMMN ALGORITHM

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ABSTRACT

In this work, a novel algorithm named sign regressor least mean mixed-norm (SRLMMN) algorithm is proposed as an alternative to the well-known least mean mixed-norm (LMMN) algorithm. The SRLMMN algorithm is a hybrid version of the sign regressor least mean square (SRLMS) and sign regressor least mean fourth (SRLMF) algorithms. Analytical expressions are derived to describe the convergence, steady-state, and tracking behavior of the proposed SRLMMN algorithm. To validate our theoretical findings, a system identification problem is considered for this purpose. It is shown that there is a very close correspondence between theory and simulation. Finally, it is also shown that the SRLMMN algorithm is robust enough in tracking the variations in the channel.

Index Terms— LMS, LMF, LMMN, SRLMS, SRLMF, SRLMMN, sign regressor, mixed-norm, convergence, steady-state, tracking.

1. INTRODUCTION

The least mean mixed-norm (LMMN) algorithm [1] is a well-known member of the family of mixed-norm adaptive filtering algorithms, which combines the benefits of the well-established least mean square (LMS) [2] and least mean fourth (LMF) algorithms [3]. In [4], the LMMN algorithm was introduced for the first time in an adaptive echo cancellation problem where it has shown improved performance over the LMS algorithm in terms of convergence and misadjustment. In-depth convergence, steady-state, and tracking analysis of the LMMN algorithm can be found in [5]–[6].

Signed adaptive filters are extensively used for the processing and analysis of electrocardiogram (ECG) signals [7] as they are computationally less complex when compared to their unsigned counterparts. The proposed sign regressor least mean mixed-norm (SRLMMN) algorithm is a signed version of the LMMN algorithm, which is obtained by taking the signum function of the input data. The SRLMMN algorithm is a hybrid algorithm based on a combination of the sign regressor least mean square (SRLMS) [8]–[9] and sign regressor least mean fourth (SRLMF) [10] algorithms. The SRLMMN algorithm reduces to SRLMF and SRLMS algorithms when the mixing parameter becomes zero and one, respectively. In the present work, analytical expressions for the convergence, steady-state mean-square error (MSE), and tracking MSE of the SRLMMN algorithm are derived.

The rest of the paper is structured as follows. The weight update equation of the proposed algorithm is described in Section 2. The convergence, steady-state, and tracking analysis of the SRLMMN algorithm is carried out in Sections 3, 4, and 5, respectively. Simulation studies which confirm the theoretical findings are presented in Section 6, followed by conclusions in Section 7.

2. THE SRLMMN ALGORITHM

The weight update equation of the LMMN algorithm can be written as follows [9]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\, \mathbf{u}_i^{\mathrm{T}} e_i[\delta + (1-\delta)e_i^2], \qquad 0 \le \delta \le 1, \tag{1}$$

where $\mathbf{w}_i \in \mathbb{R}^{M\times 1}$ is the updated weight vector at iteration i, M is the length of adaptive filter, μ is the step-size, $\mathbf{u}_i \in \mathbb{R}^{1\times M}$ is the regressor vector, δ is the mixing parameter, and e_i is the estimation error given by

$$e_i = d_i - \mathbf{u}_i\mathbf{w}_{i-1}, \tag{2}$$

where d_i is the desired value. The SRLMMN algorithm is obtained from the LMMN algorithm in (1) by replacing the regressor vector by its sign as shown below:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i[\delta + (1-\delta)e_i^2], \qquad 0 \le \delta \le 1. \tag{3}$$

3. CONVERGENCE ANALYSIS

To carry out the convergence analysis of the SRLMMN algorithm we rely on the assumptions mentioned in [11].
Subtracting both sides of (3) from the optimal weight vector $\mathbf{w}_i^o$ we get

$$\widetilde{\mathbf{w}}_i = \widetilde{\mathbf{w}}_{i-1} - \mu\,\delta\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i - \mu\,(1-\delta)\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \tag{4}$$

where the weight error vector $\widetilde{\mathbf{w}}_i$ is given by

$$\widetilde{\mathbf{w}}_i = \mathbf{w}_i^o - \mathbf{w}_i. \tag{5}$$

Taking the expectation of both sides of (4) under the assumptions mentioned in [11] we obtain

$$\mathrm{E}[\widetilde{\mathbf{w}}_i] = \mathrm{E}[\widetilde{\mathbf{w}}_{i-1}] - \mu\,\delta\,\mathrm{E}\left[\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i\right] - \mu\,(1-\delta)\mathrm{E}\left[\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3\right]. \tag{6}$$

From [10], we have

$$\mathrm{E}\left[\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i\right] = \sqrt{\frac{2}{\pi\sigma_u^2}}\mathbf{R}\mathrm{E}[\widetilde{\mathbf{w}}_{i-1}], \tag{7}$$

$$\mathrm{E}\left[\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3\right] = 3\sqrt{\frac{2}{\pi\sigma_u^2}}\sigma_e^2\mathbf{R}\mathrm{E}[\widetilde{\mathbf{w}}_{i-1}], \tag{8}$$

where σ_u^2 is the regressor variance, σ_e^2 is the estimation error variance, and $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i]$ is the regressor autocorrelation matrix. Upon substituting (7) and (8) into (6), we have

$$\mathrm{E}[\widetilde{\mathbf{w}}_i] = \left[1 - \mu\sqrt{\frac{2}{\pi\sigma_u^2}}\mathbf{R}\left(\delta + 3(1-\delta)\sigma_e^2\right)\right]\mathrm{E}[\widetilde{\mathbf{w}}_{i-1}]. \tag{9}$$

From (9), it is easy to show that the mean behavior of the weight error vector, that is $\mathrm{E}[\widetilde{\mathbf{w}}_i]$, converges to the zero vector if the step-size μ is bounded by:

$$0 < \mu < \frac{\sqrt{2\pi\sigma_u^2}}{\lambda_{\max}\left(\delta + 3(1-\delta)\sigma_e^2\right)}, \tag{10}$$

where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{R}$. We can obtain the step-size bounds of the SRLMF and SRLMS algorithms from (10) by setting δ equal to 0 and 1, respectively, as shown below:

$$0 < \mu_{\mathrm{SRLMF}} < \frac{\sqrt{2\pi\sigma_u^2}}{3\lambda_{\max}\sigma_e^2}. \tag{11}$$

$$0 < \mu_{\mathrm{SRLMS}} < \frac{\sqrt{2\pi\sigma_u^2}}{\lambda_{\max}}. \tag{12}$$

Note that the step-size bound of the SRLMF algorithm in (11) is the same as that obtained by us in [10]. Equation (10) can also be rewritten in the following equivalent form:

$$0 < \mu < \delta\,\mu_{\mathrm{SRLMS}} + (1-\delta)\mu_{\mathrm{SRLMF}}. \tag{13}$$

4. STEADY-STATE ANALYSIS

To carry out the steady-state analysis of the SRLMMN algorithm we shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the assumptions of the stationary data model mentioned in [12].

For the adaptive filter of the form in (3), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [9]:

$$\mu\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] = 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \quad \text{as} \quad i \to \infty, \tag{14}$$

where

$$\mathrm{E}[\|\mathbf{u}_i\|_{\mathrm{H}}^2] = \mathrm{E}[\mathbf{u}_i\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}], \tag{15}$$

$$e_i = e_{a_i} + v_i, \tag{16}$$

with $\mathrm{g}[e_i]$ denoting some function of e_i, and $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error. Then $\mathrm{g}[e_i]$ for the SRLMMN algorithm can be shown to be

$$\mathrm{g}[e_i] = \delta(e_{a_i} + v_i) + \bar{\delta}\{e_{a_i}^3 + e_{a_i}v_i^2 + 2e_{a_i}^2 v_i + v_i e_{a_i}^2 + v_i^3 + 2e_{a_i}v_i^2\}, \tag{17}$$

where $\bar{\delta} = 1 - \delta$. By using the fact that e_{a_i} and v_i are independent, and by ignoring third and higher-order terms of e_{a_i}, we reach at the following expression for the term $\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right]$:

$$\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right] \approx (\delta + 3\bar{\delta}\sigma_v^2)\mathrm{E}[e_{a_i}^2]. \tag{18}$$

To evaluate the term $\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2\mathrm{g}^2[e_i]\right]$, we start by noting that

$$\begin{aligned}\mathrm{g}^2[e_i] = {} & \delta^2[e_{a_i}^2 + v_i^2 + 2e_{a_i}v_i] + \bar{\delta}^2[e_{a_i}^6 + 6e_{a_i}^5 v_i + 6e_{a_i}v_i^5 \\ & + 15e_{a_i}^4 v_i^2 + 15e_{a_i}^2 v_i^4 + 20e_{a_i}^3 v_i^3 + v_i^6] + 2\delta\bar{\delta}[e_{a_i}^4 \\ & + 6e_{a_i}^2 v_i^2 + 4e_{a_i}^3 v_i + 4e_{a_i}v_i^3 + v_i^4].\end{aligned} \tag{19}$$

If we multiply $\mathrm{g}^2[e_i]$ by $\|\mathbf{u}_i\|_{\mathrm{H}}^2$ from the left, use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, and again ignoring third and higher-order terms of e_{a_i}, we obtain

$$\begin{aligned}\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2\mathrm{g}^2[e_i]\right] \approx {} & (\delta^2 + 15\bar{\delta}^2\xi_v^4 + 12\delta\bar{\delta}\sigma_v^2)\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 e_{a_i}^2\right] \\ & + (\delta^2\sigma_v^2 + \bar{\delta}^2\xi_v^6 + 2\delta\bar{\delta}\xi_v^4)\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2\right],\end{aligned} \tag{20}$$

where $\xi_v^4 = \mathrm{E}[v_i^4]$ and $\xi_v^6 = \mathrm{E}[v_i^6]$ are the fourth and sixth-order moments of the noise sequence v_i, respectively.
Substituting (18) and (20) into (14), and by using the separation principle [12], we get

$$\begin{aligned}\mu(\delta^2\sigma_v^2 + \bar{\delta}^2\xi_v^6 + 2\delta\bar{\delta}\xi_v^4)\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2\right] = {} & [2(\delta + 3\bar{\delta}\sigma_v^2) \\ & - \mu(\delta^2 + 15\bar{\delta}^2\xi_v^4 + 12\delta\bar{\delta}\sigma_v^2)\mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2\right]]\mathrm{E}[e_{a_i}^2].\end{aligned} \tag{21}$$

Ultimately, the expression for the steady-state MSE $\varphi = \mathrm{E}\left[e_i^2\right]$ of the SRLMMN algorithm can be shown to be

$$\varphi = \frac{\mu(\delta^2\sigma_v^2 + \bar{\delta}^2\xi_v^6 + 2\delta\bar{\delta}\xi_v^4)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})}{2(\delta + 3\bar{\delta}\sigma_v^2) - \mu(\delta^2 + 15\bar{\delta}^2\xi_v^4 + 12\delta\bar{\delta}\sigma_v^2)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{22}$$

We can obtain the expressions for the steady-state MSE of the SRLMF and SRLMS algorithms from (22) by setting δ equal to 0 and 1, respectively, as shown below:

$$\varphi_{\mathrm{SRLMF}} = \frac{\mu\xi_v^6\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})}{6\sigma_v^2 - 15\mu\xi_v^4\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{23}$$

$$\varphi_{\mathrm{SRLMS}} = \frac{\mu\sigma_v^2\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})}{2 - \mu\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{24}$$

Note that the expression for the steady-state MSE of the SRLMF algorithm in (23) is the same as that obtained by us in [10]. Similarly, here Equation (22) can also be rewritten in the following equivalent form:

$$\varphi = \delta\, \varphi_{\text{SRLMS}} + (1-\delta)\varphi_{\text{SRLMF}}. \tag{25}$$

5. TRACKING ANALYSIS

To carry out the tracking analysis of the SRLMMN algorithm we shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the assumptions of the nonstationary data model mentioned in [11].

For the adaptive filter of the form in (3), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [9]:

$$\mu \mathrm{E}\left[\|\mathbf{u}_i\|_{\mathrm{H}}^2 g^2[e_i]\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left[e_{a_i} g[e_i]\right], \quad \text{as } i \to \infty. \tag{26}$$

Tracking results can be obtained by inspection from the steady-state results in Section 4 as there are only minor differences. Therefore, by substituting (18) and (20) into (26) the expression for the tracking MSE φ' of the SRLMMN algorithm can be shown to be

$$\varphi' = \frac{\mu(\delta^2\sigma_v^2 + \bar{\delta}^2\xi_v^6 + 2\delta\bar{\delta}\xi_v^4)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2(\delta + 3\bar{\delta}\sigma_v^2) - \mu(\delta^2 + 15\bar{\delta}^2\xi_v^4 + 12\delta\bar{\delta}\sigma_v^2)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{27}$$

We can obtain the expressions for the tracking MSE of the SRLMF and SRLMS algorithms from (27) by setting δ equal to 0 and 1, respectively, as shown below:

$$\varphi'_{\text{SRLMF}} = \frac{\mu\xi_v^6\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{6\sigma_v^2 - 15\mu\xi_v^4\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2, \tag{28}$$

$$\varphi'_{\text{SRLMS}} = \frac{\mu\sigma_v^2\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2 - \mu\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2, \tag{29}$$

where $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$ is the autocorrelation matrix of the sequence $\mathbf{q}_i$ in the random walk model as described in the next section. Note that the expression for the tracking MSE of the SRLMF algorithm in (28) is the same as that obtained by us in [10]. Similarly, here Equation (27) can also be rewritten in the following equivalent form:

$$\varphi' = \delta\, \varphi'_{\text{SRLMS}} + (1-\delta)\varphi'_{\text{SRLMF}}. \tag{30}$$

6. SIMULATION RESULTS

In all the simulations, the problem of identification of an unknown system is considered with filter tap-length of $M = 10$. The tap-weight vector of the unknown system is considered to be stationary for both convergence and steady-state MSE of the SRLMMN algorithm and nonstationary for tracking MSE of the SRLMMN algorithm. The amount of nonstationarity added to the tap-weight vector is according to a random walk model, as described later in this section. The signal-to-noise ratio (SNR) is fixed at 30 dB (Figures 1–4), 20 dB (Figures 5–8), and 10 dB (Figures 9–10). The mixing parameter is fixed at 0.5 for Figures 1, 5, and 9. The step-size is fixed at 0.005 for Figure 2 and 0.01 for Figure 6. We have considered additive white Gaussian noise (AWGN) environment for Figures 1–8 and uniform noise environment for Figures 9–10.

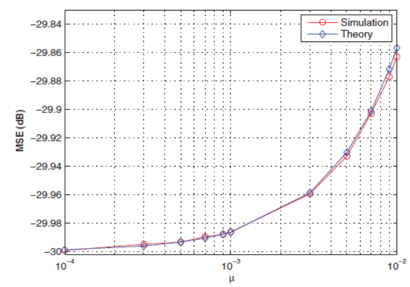

Fig. 1. Steady-state behavior of the SRLMMN algorithm versus μ for fixed δ ($\delta = 0.5$).

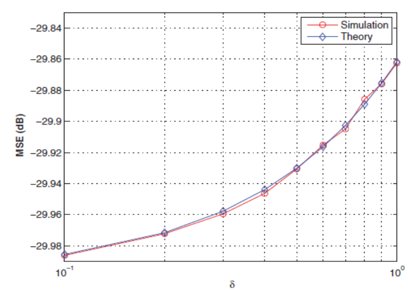

Fig. 2. Steady-state behavior of the SRLMMN algorithm versus δ for fixed μ ($\mu = 0.005$).

To demonstrate the tracking abilities of the SRLMMN algorithm, we have considered a random walk model as shown below:

$$\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i, \tag{31}$$

where $\mathbf{q}_i$ is a zero mean Gaussian noise sequence with variance $\sigma_q^2 = 10^{-10}$.

In order to examine the steady-state and tracking behavior of the SRLMMN algorithm, we have performed simulations for four different cases: In the first case, steady-state/tracking MSE of the SRLMMN algorithm is plotted against the step-size for a fixed value of the mixing parameter, as shown in

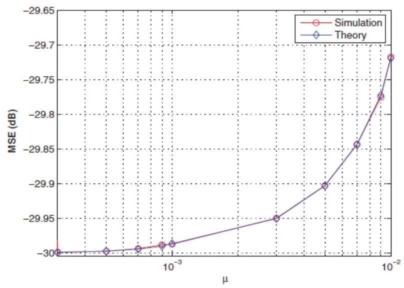

Fig. 3. Steady-state behavior of the SRLMMN algorithm versus μ for varying δ (δ varying from 0 to 1).

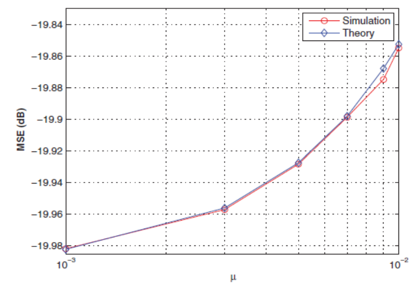

Fig. 5. Tracking behavior of the SRLMMN algorithm versus μ for fixed δ (δ = 0.5).

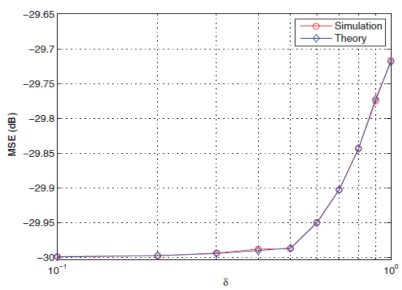

Fig. 4. Steady-state behavior of the SRLMMN algorithm versus δ for varying μ (μ varying from 10^{-4} to 10^{-2}).

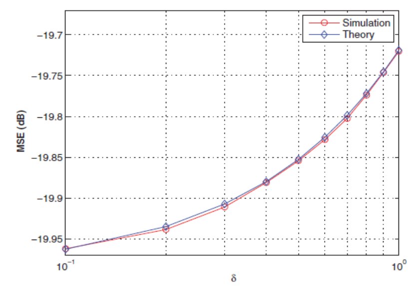

Fig. 6. Tracking behavior of the SRLMMN algorithm versus δ for fixed μ (μ = 0.01).

Figures 1 and 5, respectively. In the second case, steady-state/tracking MSE of the SRLMMN algorithm is plotted against the mixing parameter for a fixed value of the step-size, as shown in Figures 2 and 6, respectively. In the third case, steady-state/tracking MSE of the SRLMMN algorithm is plotted against the step-size for varying values of the mixing parameter, as shown in Figures 3 and 7, respectively. Finally, in the fourth case, steady-state/tracking MSE of the SRLMMN algorithm is plotted against the mixing parameter for varying values of the step-size, as shown in Figures 4 and 8, respectively. In all these cases, an excellent match is observed between the theoretical and simulated results, as shown in Figures 1–8.

Moreover, in order to study the convergence aspects of the SRLMMN algorithm some results are presented in Figures 9–10. Figure 9 compares the convergence rate of the LMMN and SRLMMN algorithms. We can observe from Figure 9 that the learning curves of both the algorithms are almost identical. Finally, Figure 10 compares the convergence rate of the SRLMMN algorithm for various values of δ. Notice in Figure 10 that, the SRLMMN algorithm reduces to SRLMF and SRLMS algorithms when $\delta = 0$ and $\delta = 1$, respectively.

7. CONCLUSIONS

A new variant of the LMMN algorithm called the SRLMMN algorithm has been introduced and analyzed in this paper. It has been shown that the SRLMMN algorithm can achieve similar performance with respect to its adaptive counterpart but with reduced complexity. Furthermore, an excellent agreement is observed between the simulation and analytical results. Thus, a combination of the SRLMS and SRLMF algorithms has resulted in a hybrid algorithm, which is robust to variations in the channel statistics.

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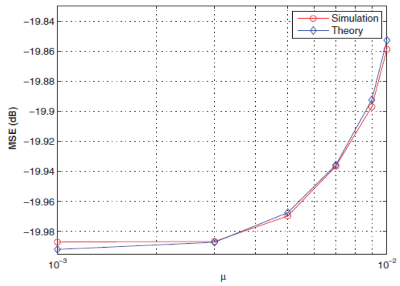

Fig. 7. Tracking behavior of the SRLMMN algorithm versus μ for varying δ (δ varying from 0 to 1).

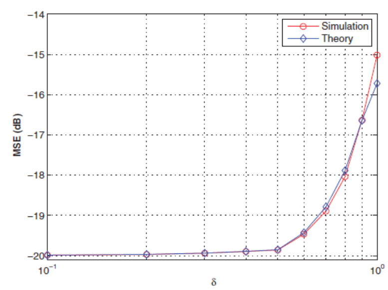

Fig. 8. Tracking behavior of the SRLMMN algorithm versus δ for varying μ (μ varying from 10^{-3} to 10^{-1}).

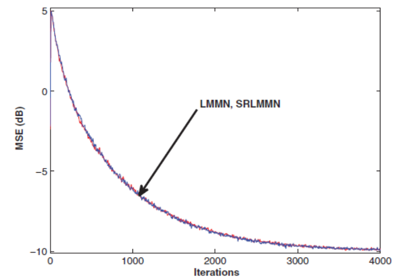

Fig. 9. Behavior of the LMMN and SRLMMN algorithms.

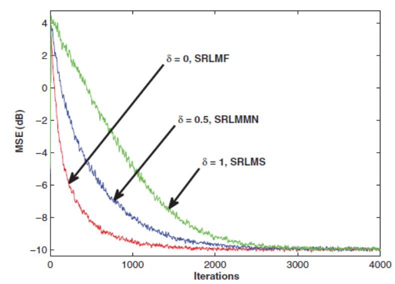

Fig. 10. Behavior of the SRLMMN algorithm for different values of δ.

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Removal of multiple artifacts from ECG signal using cascaded multistage adaptive noise cancellers

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ABSTRACT

Although cascaded multistage adaptive noise cancellers have been employed before by researchers for multiple artifact removal from the ElectroCardioGram (ECG) signal, they all used the same adaptive algorithm in all the cascaded multi-stages for adjusting the adaptive filter weights. In this paper, we propose a cascaded 4-stage adaptive noise canceller for the removal of four artifacts present in the ECG signal, viz. baseline wander, motion artifacts, muscle artifacts, and 60 Hz Power Line Interference (PLI). We have investigated the performance of eight adaptive algorithms, viz. Least Mean Square (LMS), Least Mean Fourth (LMF), Least Mean Mixed-Norm (LMMN), Sign Regressor Least Mean Square (SRLMS), Sign Error Least Mean Square (SELMS), Sign-Sign Least Mean Square (SSLMS), Sign Regressor Least Mean Fourth (SRLMF), and Sign Regressor Least Mean Mixed-Norm (SRLMMN) in terms of Signal-to-Noise Ratio (SNR) improvement for removing the aforementioned four artifacts from the ECG signal. We employed the LMMN, LMF, LMMN, LMF algorithms in the proposed cascaded 4-stage adaptive noise canceller to remove the respective ECG artifacts as mentioned above. We succeeded in achieving an SNR improvement of 12.7319 dBs. The proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF algorithms outperforms those that employ the same algorithm in the four stages. One unique and powerful feature of our proposed cascaded 4-stage adaptive noise canceller is that it employs only those adaptive algorithms in the four stages, which are shown to be effective in removing the respective ECG artifacts as mentioned above. Such a scheme has not been investigated before in the literature.

1. Introduction

Adaptive noise cancellation is a method of estimating signals, which are corrupted by additive noise or interference. This method employs a primary input, which is the corrupted signal, and a secondary or reference input, which is the noise correlated with the noise present in the primary input. The reference input is adaptively filtered and subtracted from the primary input in order to obtain the signal estimate. The adaptive noise cancellation method can be employed whenever an appropriate reference input is available [1,2].

Thakor and Zhu [3] proposed several adaptive filter structures for noise cancellation and arrhythmia detection in ECG signals. The diverse forms of noise like baseline wander, 60 Hz PLI, muscle artifacts, and motion artifacts were eliminated from the ECG signal [3]. Hamilton [4] investigated the relative performance of an adaptive and nonadaptive 60-Hz notch filters for the reduction of PLI in the ECG signal. Ziarani and Konrad [5] proposed a nonlinear adaptive method of elimination of PLI from the ECG signal. The proposed method offered a robust structure and is shown to have a high degree of immunity with respect to external noise [5]. Raya and Sison [6] proposed an adaptive noise cancellation method to remove motion artifacts in stress ECG signals by using an accelerometer. The adaptive noise cancellers in [6] are implemented using the two of the most widely employed adaptive filtering algorithms, viz. LMS and Recursive Least Squares (RLS). Martens et al. [7] proposed an improved adaptive noise canceller for the reduction of the fundamental PLI component and harmonics in the ECG signal. Behbahani [8] simulated and tested an adaptive noise cancellation method using the LMS algorithm for removing the 60 Hz PLI. Lin and Hu [9] developed an efficient RLS adaptive notch filter for the suppression of PLI in the ECG signal. They also proposed a PLI detector that employed an optimal linear discriminant analysis algorithm for the detection of PLI in the ECG signal [9].

Rahman et al. [10–12, range] employed Normalized Sign Regressor Least Mean Square (NSRLMS), Normalized Sign Error Least Mean Square (NSELMS), and Normalized Sign-Sign Least Mean Square (NSSLMS) algorithms for canceling various artifacts such as baseline wander, 60 Hz PLI, muscle artifacts, and motion artifacts from the ECG signal. Rahman et al. [13] employed LMS, SRLMS, SELMS, and SSLMS algorithms for canceling various artifacts as mentioned

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above from the ECG signal. In [13], it is shown that the performance of the SRLMS algorithm is superior to the LMS algorithm in terms of SNR improvement. Rahman et al. [14] expanded the work in [10–12, range] by employing Block-Based Normalized Sign Regressor Least Mean Square (BBNSRLMS), Block-Based Normalized Sign Error Least Mean Square (BBNSELMS), and Block-Based Normalized Sign-Sign Least Mean Square (BBNSSLMS) algorithms for canceling various artifacts as mentioned above from the ECG signal.

Islam et al. [15] added the four types of Alternating Current (AC) and Direct Current (DC) interference/noise with ECG signals and nullified these noises using the LMS and RLS algorithms. Vullings et al. [16] developed an adaptive Kalman filter to enhance the quality of the ECG signal. Dhubkarya et al. [17] implemented an adaptive noise canceller for denoising an ECG signal and tested the performance of the system using various algorithms such as LMS, Normalized Least Mean Square (NLMS), and RLS. Chandrakar and Kowar [18] employed the RLS algorithm for the removal of different kinds of noises from the ECG signal. Kim et al. [19] proposed a motion artifact removal method using a cascaded 2-stage LMS adaptive filter for an ambulatory ECG monitoring system. Mugdha et al. [20] conducted a study of the RLS algorithm in noise removal from ECG signals and concluded that the RLS algorithm is more efficient in removing noises from ECG signals than the LMS algorithm.

Ebrahimzadeh et al. [21] compared various kinds of ECG noise reduction algorithms such as LMS, Block-Based Least Mean Square (BBLMS), NLMS, Unbiased and Normalized Adaptive Noise Reduction (UNANR), and RLS. Sharma et al. [22] used an adaptive noise canceller that employs LMS algorithm for ECG noise removal and concluded that an increase in the step-size increases the noise as well as the rate of convergence. Satheeskumaran and Sabrigiriraj [23] proposed a Variable Step Size Delayed Least Mean Square (VSSDLMS) adaptive filter to remove the artifacts from the ECG signal. Sehamby and Singh [24] used an LMS-based adaptive noise canceller to derive a noise-free fetal ECG signal. Haritha et al. [25] surveyed different filters and denoising techniques used for ECG signals. Qureshi et al. [26] proposed a cascaded 3-stage adaptive noise canceller to eliminate three types of artifacts from the ECG signal, viz. baseline wander, 60 Hz PLI, and motion artifacts. The same algorithm was used in all three stages of the cascaded adaptive noise canceller. The results of a cascaded 3-stage LMS-based adaptive noise canceller were compared with those of a cascaded 3-stage NLMS-based adaptive noise canceller, a cascaded 3-stage Log LMS-based adaptive noise canceller, and a cascaded 3-stage SRLMS-based adaptive noise canceller. Warmerdam et al. [27] proposed a fixed-lag Kalman smoother to filter PLI from ECG recordings with minimal distortion of the ECG waveform.

Sutha and Jayanthi [28] discuss prototype hardware developed to monitor and record the raw mother ECG signal containing the fetal ECG and a signal processing algorithm to extract the fetal ECG. The adaptive noise canceller employed in their work uses the SSLMS algorithm [28]. Gilani et al. [29] employed an LMS-based adaptive noise canceller to remove the 50 Hz PLI from the ECG signal. Venkatesan et al. [30] studied a Delayed Error Normalized Least Mean Square (DENLMS) adaptive filter with pipelined architecture to remove the white Gaussian noise from the ECG signal. Srinivasa and Pandian [31] eliminate the 50 Hz PLI from ECG signal using an LMS-based adaptive noise canceller. Xiong et al. [32] have shown that the cosine-based adaptive algorithm is superior to the standard LMS algorithm in reducing the high amplitude motion artifact noise from the ECG signal. Saxena et al. [33] remove the 50 Hz PLI from the ECG signal using an NLMS-based adaptive noise canceller. Manju and Sneha [34] performed ECG denoising using Weiner filter and Kalman filter. Their results have shown that the Wiener filter performs better than the Kalman filter for ECG noise removal. Khiter et al. [35] proposed a novel adaptive denoising method called self correcting leaky normalized least mean square algorithm with varied step size and leakage coefficient for reducing the muscle artifacts from the ECG signal. Yadav et al. [36] applied the symbiotic organisms search algorithm for estimating the weight vectors of an optimized adaptive noise canceller for reducing the artifacts from the ECG signal.

In this paper, we will employ a cascaded 4-stage adaptive noise canceller to remove the four types of artifacts from the ECG signal, viz. baseline wander, motion artifacts, muscle artifacts, and 60 Hz PLI. The contributions of this paper are: (1) We first determine the best performing adaptive algorithms in terms of SNR improvement among the eight adaptive algorithms studied in this paper, viz. Least Mean Square (LMS), Least Mean Fourth (LMF), Least Mean Mixed-Norm (LMMN), Sign Regressor Least Mean Square (SRLMS), Sign Error Least Mean Square (SELMS), Sign-Sign Least Mean Square (SSLMS), Sign Regressor Least Mean Fourth (SRLMF), and Sign Regressor Least Mean Mixed-Norm (SRLMMN) for removing the aforementioned four artifacts from the ECG signal, (2) We then employ the four shortlisted algorithms, viz. LMMN, LMF, LMMN, LMF in the proposed cascaded 4-stage adaptive noise canceller for removing the aforementioned four artifacts from the ECG signal, and (3) We then compare the performance of the proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF algorithms with those that employ the LMS, LMS, LMS, LMS algorithms, the LMF, LMF, LMF, LMF algorithms, the LMMN, LMMN, LMMN, LMMN algorithms, and the SRLMMN, SRLMF, SRLMMN, SRLMF algorithms. We were able to achieve a significant improvement in the SNR of the filtered ECG signal after the application of our proposed scheme over other schemes. The remainder of this paper is organized as follows. Various adaptive algorithms studied in this paper are discussed in Section 2. The proposed cascaded 4-stage adaptive noise canceller is discussed in Section 3. Simulation results are discussed in Section 4. Finally, the paper is concluded in Section 5.

2. Adaptive algorithms

In this work, we have studied eight adaptive algorithms, viz. LMS, LMF, LMMN, SRLMS, SELMS, SSLMS, SRLMF, and SRLMMN for the removal of multiple artifacts present in the ECG signal. The weight update equations of these eight adaptive algorithms are given in Table 1 wherein $\mathbf{w}_i \in \mathbb{R}^{M\times 1}$ is the updated weight vector at iteration $i \geq 0$, M is the adaptive filter length, μ is the step-size, $\mathbf{u}_i \in \mathbb{R}^{1\times M}$ is the regressor or input vector with variance σ_u^2, δ is the mixing parameter ranging between $0 \leq \delta \leq 1$, e_i is the estimation error given by

$$e_i = d_i - \mathbf{u}_i\mathbf{w}_{i-1}, \tag{1}$$

where d_i is the desired value, and

$$\mathrm{sgn}[x] = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases} \tag{2}$$

The LMMN algorithm is a combination of the LMS and LMF algorithms as long as the mixing parameter is ranging between $0 < \delta < 1$. The LMMN algorithm reduces to LMF and LMS algorithms when the mixing parameter becomes zero and one, respectively.

The sign adaptive filters are used for the processing and analysis of ECG signals as they are computationally less complex. However, the performance of a sign adaptive filter is compromised because of the clipping effect due to the application of signum function to either the regressor vector, estimation error, or both. The SRLMS, SELMS, and SSLMS algorithms are also known in the literature as the Sign Regressor Algorithm (SRA), Sign Algorithm (SA), and Sign-Sign Algorithm (SSA), respectively. The SRLMMN algorithm is a combination of the SRLMS and SRLMF algorithms as long as the mixing parameter is ranging between $0 < \delta < 1$. The SRLMMN algorithm reduces to SRLMF and SRLMS algorithms when the mixing parameter becomes zero and one, respectively. Note that the SRLMF [37] and SRLMMN [38] algorithms were developed by us and are being employed in this work for the removal of multiple artifacts present in the ECG signal.

Table 1
Weight update equations of various adaptive algorithms.

Adaptive algorithm	Weight update equation
LMS [39,40]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathbf{u}_i^{\mathrm{T}} e_i$
LMF [40,41]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathbf{u}_i^{\mathrm{T}} e_i^3$
LMMN [42]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathbf{u}_i^{\mathrm{T}} e_i[\delta + (1-\delta)e_i^2]$
SRLMS [43]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathrm{sgn}[\mathbf{u}_i]^{\mathrm{T}} e_i$
SELMS [44]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathbf{u}_i^{\mathrm{T}} \mathrm{sgn}[e_i]$
SSLMS [45]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathrm{sgn}[\mathbf{u}_i]^{\mathrm{T}} \mathrm{sgn}[e_i]$
SRLMF [37]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathrm{sgn}[\mathbf{u}_i]^{\mathrm{T}} e_i^3$
SRLMMN [38]	$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathrm{sgn}[\mathbf{u}_i]^{\mathrm{T}} e_i[\delta + (1-\delta)e_i^2]$

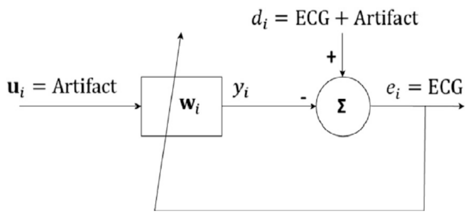

Fig. 1. Adaptive noise canceller.

3. Proposed cascaded 4-stage adaptive noise canceller

A single-stage adaptive noise canceller for removing a single artifact from the ECG signal is shown in Fig. 1. As can be seen from this figure d_i forms the primary input of the adaptive noise canceller, d_i contains the ECG signal with an additive artifact, $\mathbf{u}_i$ forms the secondary or reference input of the adaptive noise canceller, $\mathbf{u}_i$ contains the reference artifact that is correlated only with the artifact present in the corrupted ECG signal d_i, $\mathbf{w}_i$ are the adaptive filter coefficients, y_i is the adaptive filter output, and e_i is the filtered ECG signal free from the artifact.

A proposed cascaded 4-stage adaptive noise canceller for removing the four artifacts from the ECG signal is shown in Fig. 2. As can be seen from this figure d_{i1} forms the primary input of the first adaptive noise canceller, d_{i1} contains the ECG signal with four additive artifacts, viz. baseline wander, motion artifacts, muscle artifacts, and 60 Hz PLI, $\mathbf{u}_{i1}$ forms the secondary or reference input of the first adaptive noise canceller, $\mathbf{u}_{i1}$ contains the reference baseline wander that is correlated only with the baseline wander present in the corrupted ECG signal d_{i1}, $\mathbf{u}_{i2}$ forms the secondary or reference input of the second adaptive noise canceller, $\mathbf{u}_{i2}$ contains the reference motion artifacts that is correlated only with the motion artifacts present in the corrupted ECG signal d_{i1}, $\mathbf{u}_{i3}$ forms the secondary or reference input of the third adaptive noise canceller, $\mathbf{u}_{i3}$ contains the reference muscle artifacts that is correlated only with the muscle artifacts present in the corrupted ECG signal d_{i1}, $\mathbf{u}_{i4}$ forms the secondary or reference input of the fourth adaptive noise canceller, $\mathbf{u}_{i4}$ contains the reference 60 Hz PLI that is correlated only with the 60 Hz PLI present in the corrupted ECG signal d_{i1}, $\mathbf{w}_{i1}$ to $\mathbf{w}_{i4}$ are the respective adaptive filter coefficients, y_{i1} to y_{i4} are the respective adaptive filter outputs, e_{i1} is the partially corrupted ECG signal free from baseline wander, e_{i1} will act as the primary input d_{i2} to the second adaptive noise canceller, e_{i2} is the partially corrupted ECG signal free from baseline wander and motion artifacts, e_{i2} will act as the primary input d_{i3} to the third adaptive noise canceller, e_{i3} is the partially corrupted ECG signal free from baseline wander, motion artifacts, and muscle artifacts, e_{i3} will act as the primary input d_{i4} to the fourth adaptive noise canceller and e_{i4} is the filtered ECG signal free from baseline wander, motion artifacts, muscle artifacts, and 60 Hz PLI. One unique and powerful feature of our proposed cascaded 4-stage adaptive noise canceller is that it employs only those adaptive algorithms in the four stages, which are shown to be effective in the subsequent section in removing the aforementioned four artifacts from the ECG signal.

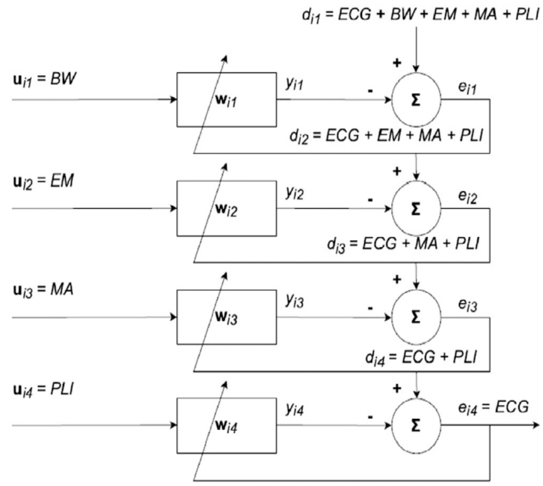

Fig. 2. Proposed cascaded 4-stage adaptive noise canceller.

4. Simulation results

4.1. Baseline wander removal

In this experiment, the step-size is fixed at $\mu = 0.01$, the adaptive filter length is fixed at $M = 5$, the noise variance is fixed at $\sigma_v^2 = 0.1$, and the number of iterations is fixed at $L = 10$ for all the eight adaptive algorithms studied. In addition to the above settings, the mixing parameter is fixed at $\delta = 0.5$ for the LMMN and SRLMMN algorithms.

In this case, 3600 samples of the clean ECG signal are taken from the MIT-BIH Arrhythmia Database (MITDB) Record: 105 [46], and they are later added with the 3600 samples of baseline wander taken from the MIT-BIH Noise Stress Test Database (NSTDB) Record: bw [46].

All eight adaptive algorithms studied in this paper, viz. LMS, LMF, LMMN, SRLMS, SELMS, SSLMS, SRLMF, and SRLMMN are tested separately by plugging them in a single-stage adaptive noise canceller as described in Fig. 1 for baseline wander removal. The SNR before and after adaptive filtering is recorded in Table 2. The SNR is calculated by using the built-in MATLAB function, viz. $snr(x, y)$. The SNR before and after adaptive filtering in Table 2 is calculated as described by the MATLAB code fragment in Appendix A. Here, y is the adaptive filter output. Note that the ECG signal and baseline wander have a gain of 200 each. Therefore, we divide these signals by 200 as shown in the MATLAB code fragment in Appendix A.

As can be seen from Table 2 the LMMN algorithm outperforms the other seven algorithms in terms of SNR improvement. The Mean Square Error (MSE) plot after baseline wander removal using a single-stage adaptive noise canceller employing the SSLMS adaptive algorithm, which is the worst-case scenario among the eight algorithms studied is shown in Fig. 3.

4.2. Motion artifacts removal

In this experiment, the step-size is fixed at $\mu = 0.01$, the adaptive filter length is fixed at $M = 5$, the noise variance is fixed at $\sigma_v^2 = 0.1$, and the number of iterations is fixed at $L = 10$ for all the eight adaptive algorithms studied. In addition to the above settings, the mixing parameter is fixed at $\delta = 0.5$ for the LMMN and SRLMMN algorithms.

In this case, 3600 samples of the clean ECG signal are taken from the MIT-BIH Arrhythmia Database (MITDB) Record: 105 [46], and they

Table 2
Baseline wander removal using a single-stage adaptive noise canceller.

Adaptive algorithm	SNR before filtering (dB)	SNR after filtering (dB)	SNR improvement (dB)
LMS	7.9251	7.9446	0.0195
LMF	7.9251	7.9513	0.0262
LMMN	7.9251	8.9812	1.0561
SRLMS	7.9251	3.3297	−4.5954
SELMS	7.9251	3.9091	−4.0160
SSLMS	7.9251	1.1036	−6.8215
SRLMF	7.9251	8.2505	0.3254
SRLMMN	7.9251	5.2039	−2.7212

Table 3
Motion artifacts removal using a single-stage adaptive noise canceller.

Adaptive algorithm	SNR before filtering (dB)	SNR after filtering (dB)	SNR improvement (dB)
LMS	5.7109	3.7061	−2.0048
LMF	5.7109	5.7874	0.0765
LMMN	5.7109	4.4862	−1.2247
SRLMS	5.7109	2.1133	−3.5976
SELMS	5.7109	1.4867	−4.2242
SSLMS	5.7109	0.6071	−5.1038
SRLMF	5.7109	4.0887	−1.6222
SRLMMN	5.7109	2.7931	−2.9178

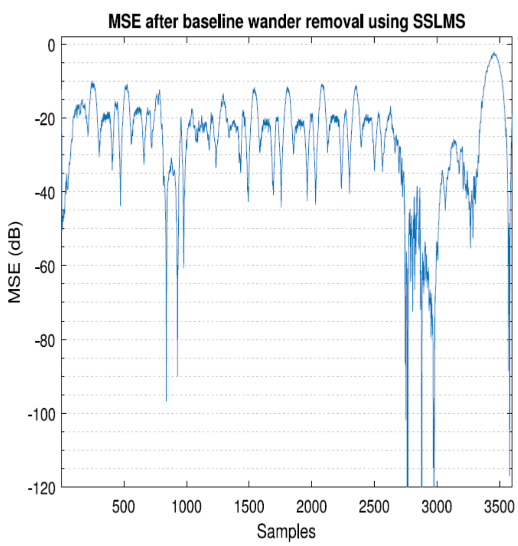

Fig. 3. MSE after baseline wander removal using a single-stage adaptive noise canceller employing the SSLMS adaptive algorithm (worst-case scenario)

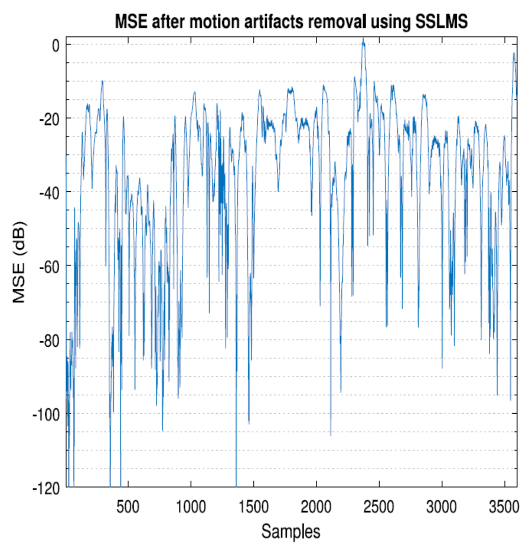

Fig. 4. MSE after motion artifacts removal using a single-stage adaptive noise canceller employing the SSLMS adaptive algorithm (worst-case scenario).

are later added with the 3600 samples of motion artifacts taken from the MIT-BIH Noise Stress Test Database (NSTDB) Record: em [46].

All eight adaptive algorithms studied in this paper, viz. LMS, LMF, LMMN, SRLMS, SELMS, SSLMS, SRLMF, and SRLMMN are tested separately by plugging them in a single-stage adaptive noise canceller as described in Fig. 1 for motion artifacts removal. The SNR before and after adaptive filtering is recorded in Table 3. The SNR before and after adaptive filtering in Table 3 is calculated by replacing line five in Appendix A MATLAB code fragment with $load('emm')$; Note that the motion artifacts have a gain of 200. Therefore, we divide this signal by 200 as shown in the MATLAB code fragment in Appendix A. As can be seen from Table 3 the LMF algorithm outperforms the other seven algorithms in terms of SNR improvement. The MSE plot after motion artifacts removal using a single-stage adaptive noise canceller employing the SSLMS adaptive algorithm, which is the worst-case scenario among the eight algorithms studied is shown in Fig. 4.

4.3. Muscle artifacts removal

In this experiment, the step-size is fixed at $\mu = 0.01$, the adaptive filter length is fixed at $M = 5$, the noise variance is fixed at $\sigma_v^2 = 0.1$, and the number of iterations is fixed at $L = 100$ for all the eight adaptive algorithms studied. In addition to the above settings, the mixing parameter is fixed at $\delta = 0.5$ for the LMMN and SRLMMN algorithms.

In this case, 3600 samples of the clean ECG signal are taken from the MIT-BIH Arrhythmia Database (MITDB) Record: 105 [46], and they are later added with the 3600 samples of muscle artifacts taken from the MIT-BIH Noise Stress Test Database (NSTDB) Record: ma [46].

All eight adaptive algorithms studied in this paper, viz. LMS, LMF, LMMN, SRLMS, SELMS, SSLMS, SRLMF, and SRLMMN are tested separately by plugging them in a single-stage adaptive noise canceller as described in Fig. 1 for muscle artifacts removal. The SNR before and after adaptive filtering is recorded in Table 4. The SNR before and

Table 4
Muscle artifacts removal using a single-stage adaptive noise canceller.

Adaptive algorithm	SNR before filtering (dB)	SNR after filtering (dB)	SNR improvement (dB)
LMS	17.8230	23.8256	6.0026
LMF	17.8230	21.0251	3.2021
LMMN	17.8230	26.1239	8.3009
SRLMS	17.8230	10.0358	−7.7872
SELMS	17.8230	19.2611	1.4381
SSLMS	17.8230	5.4269	−12.3961
SRLMF	17.8230	16.5538	−1.2692
SRLMMN	17.8230	12.3562	−5.4668

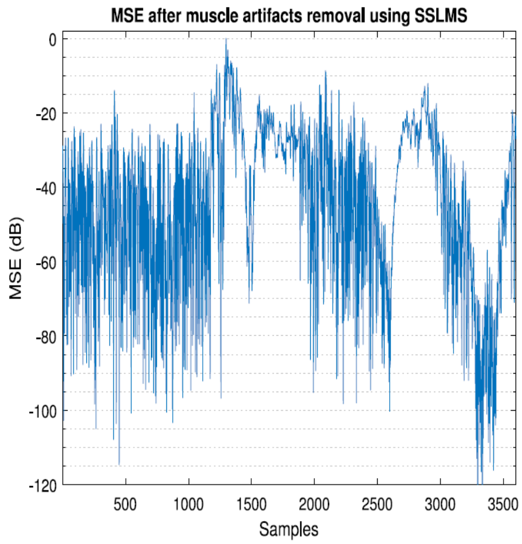

Fig. 5. MSE after muscle artifacts removal using a single-stage adaptive noise canceller employing the SSLMS adaptive algorithm (worst-case scenario).

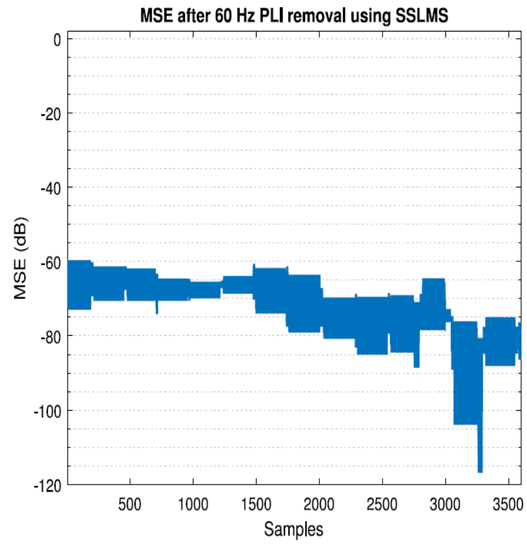

Fig. 6. MSE after 60 Hz PLI removal using a single-stage adaptive noise canceller employing the SSLMS adaptive algorithm (worst-case scenario).

after adaptive filtering in Table 4 is calculated by replacing line five in Appendix A MATLAB code fragment with $load('mam')$; Note that the muscle artifacts have a gain of 200. Therefore, we divide this signal by 200 as shown in the MATLAB code fragment in Appendix A. As can be seen from Table 4 the LMMN algorithm outperforms the other seven algorithms in terms of SNR improvement. The MSE plot after muscle artifacts removal using a single-stage adaptive noise canceller employing the SSLMS adaptive algorithm, which is the worst-case scenario among the eight algorithms studied is shown in Fig. 5.

4.4. 60 Hz PLI removal

In this experiment, the step-size is fixed at $\mu = 0.01$, the adaptive filter length is fixed at $M = 5$, and the number of iterations is fixed at $L = 10$ for all the eight adaptive algorithms studied. In addition to the above settings, the mixing parameter is fixed at $\delta = 0.5$ for the LMMN and SRLMMN algorithms.

In this case, 3600 samples of the clean ECG signal are taken from the MIT-BIH Arrhythmia Database (MITDB) Record: 105 [46], and they are later added with the 3600 samples of synthetic PLI with amplitude 100 mV, frequency 60 Hz, and sampled at 360 Hz, which has been chosen to be the same as the rest of the ECG signals used throughout our experiments.

All eight adaptive algorithms studied in this paper, viz. LMS, LMF, LMMN, SRLMS, SELMS, SSLMS, SRLMF, and SRLMMN are tested separately by plugging them in a single-stage adaptive noise canceller as described in Fig. 1 for the 60 Hz PLI removal. The SNR before and after adaptive filtering is recorded in Table 5. The SNR before and after adaptive filtering in Table 5 is calculated as described by the MATLAB code fragment in Appendix B. Here, y is the adaptive filter output. Note that the ECG signal has a gain of 200. Therefore, we divide this signal by 200 as shown in the MATLAB code fragment in Appendix B.

As can be seen from Table 5 the LMF algorithm outperforms the other seven algorithms in terms of SNR improvement. The MSE plot after 60 Hz PLI removal using a single-stage adaptive noise canceller employing the SSLMS adaptive algorithm, which is the worst-case scenario among the eight algorithms studied is shown in Fig. 6.

4.5. Multiple artifacts removal

In this experiment, the step-size is fixed at $\mu = 0.01$, the adaptive filter length is fixed at $M = 5$, the noise variance is fixed at $\sigma_v^2 = 0.1$, and the number of iterations is fixed at $L = 10$ for all the algorithms presented in Table 6. In addition to the above settings, the mixing parameter is fixed at $\delta = 0.5$ for the LMMN and SRLMMN algorithms.

In this case, 3600 samples of the clean ECG signal are taken from the MIT-BIH Arrhythmia Database (MITDB) Record: 105 [46], and they are later added with the 3600 samples of baseline wander taken from the MIT-BIH Noise Stress Test Database (NSTDB) Record: bw [46], the 3600 samples of motion artifacts taken from the MIT-BIH Noise Stress Test Database (NSTDB) Record: em [46], the 3600 samples of muscle artifacts taken from the MIT-BIH Noise Stress Test Database (NSTDB) Record: ma [46], and the 3600 samples of synthetic PLI with amplitude 100 mV, frequency 60 Hz, and sampled at 360 Hz.

The four adaptive algorithms, viz. LMMN, LMF, LMMN, and LMF shortlisted from the four experiments as discussed in Sections 4.1–4.4 are tested by plugging them in the proposed cascaded 4-stage

Table 5
60 Hz PLI removal using a single-stage adaptive noise canceller.

Adaptive algorithm	SNR before filtering (dB)	SNR after filtering (dB)	SNR improvement (dB)
LMS	14.6914	14.2872	−0.4042
LMF	14.6914	16.4652	1.7738
LMMN	14.6914	15.3068	0.6154
SRLMS	14.6914	14.1104	−0.5810
SELMS	14.6914	16.0296	1.3382
SSLMS	14.6914	13.6714	−1.0200
SRLMF	14.6914	15.2992	0.6078
SRLMMN	14.6914	14.2847	−0.4067

adaptive noise canceller as described in Fig. 2 for removing baseline wander, motion artifacts, muscle artifacts, and 60 Hz PLI from the ECG signal, respectively. We then compare the performance of the proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF algorithms with that employing the LMS, LMS, LMS, LMS algorithms, the LMF, LMF, LMF, LMF algorithms, the LMMN, LMMN, LMMN, LMMN algorithms, and the SRLMMN, SRLMF, SRLMMN, SRLMF algorithms. The SNR before and after adaptive filtering is recorded in Table 6. As can be seen from this table, we have achieved a significant improvement in the SNR by employing the LMMN, LMF, LMMN, LMF algorithms in the proposed cascaded 4-stage adaptive noise canceller. The SNR before and after adaptive filtering in Table 6 is calculated as described by the MATLAB code fragment in Appendix C. Here, y is the adaptive filter output. Note that the ECG signal, baseline wander, motion artifacts, and muscle artifacts have a gain of 200 each. Therefore as before, we divide these signals by 200 as shown in the MATLAB code fragment in Appendix C.

As an example, in row 2 of Table 6, the LMMN algorithm is used in adaptive noise cancellers 1 and 3 in Fig. 2 for removing baseline wander and muscle artifacts, respectively. The LMF algorithm in row 2 of Table 6 is used in adaptive noise cancellers 2 and 4 in Fig. 2 for removing motion artifacts and 60 Hz PLI, respectively. The MSE plot after multiple artifacts removal using the proposed cascaded 4-stage adaptive noise canceller employing the SRLMMN, SRLMF, SRLMMN, SRLMF algorithms, which is the worst-case scenario among the algorithms studied in Table 6 is shown in Fig. 7. The MSE plot after multiple artifacts removal using the proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF algorithms, which is the best-case scenario among the algorithms studied in Table 6 is shown in Fig. 8. Figs. 9(a) and 10(d) show the clean ECG signal free from artifacts, Figs. 9(b) and 10(e) show the ECG signal with additive baseline wander, motion artifacts, muscle artifacts, and 60 Hz PLI, Fig. 9(c) shows the filtered ECG signal from the proposed cascaded 4-stage adaptive noise canceller employing the SRLMMN, SRLMF, SRLMMN, SRLMF algorithms for multiple artifacts removal, which is the worst-case scenario among the algorithms studied in Table 6, and Fig. 10(f) shows the filtered ECG signal from the proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF algorithms for multiple artifacts removal, which is the best-case scenario among the algorithms studied in Table 6. As can be seen from Fig. 10(f) the LMMN, LMF, LMMN, LMF algorithms are found to be effective in removing the respective multiple artifacts from the ECG signal demonstrating our proposed scheme outperforms those in the open literature, which primarily concentrate on LMS. It is worth noting that the last three schemes in Table 6, viz. the LMF, LMF, LMF, LMF algorithms, the LMMN, LMMN, LMMN, LMMN algorithms, and the SRLMMN, SRLMF, SRLMMN, SRLMF algorithms have also not been tested before in the literature.

5. Conclusions

From our experiments, we have found that the LMMN algorithm is best suited for removing the baseline wander and muscle artifacts and the LMF algorithm is best suited for removing the motion artifacts and 60 Hz PLI. We employed the LMMN, LMF, LMMN, LMF

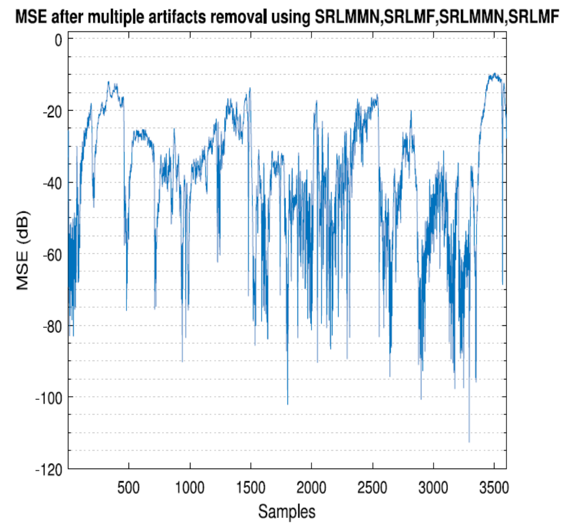

Fig. 7. MSE after multiple artifacts removal using the proposed cascaded 4-stage adaptive noise canceller employing the SRLMMN, SRLMF, SRLMMN, SRLMF adaptive algorithms (worst-case scenario).

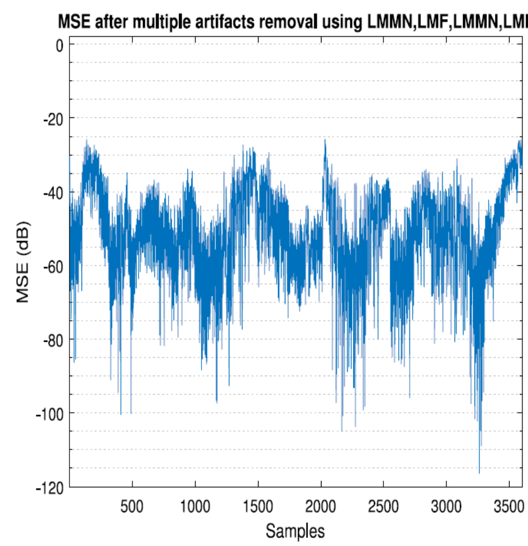

Fig. 8. MSE after multiple artifacts removal using the proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF adaptive algorithms (best-case scenario).

algorithms in the proposed cascaded 4-stage adaptive noise canceller

Table 6
Multiple ECG artifacts (Baseline Wander, Motion, Muscle, 60 Hz PLI) removal using the proposed cascaded 4-stage adaptive noise canceller.

Adaptive algorithm	SNR before filtering (dB)	SNR after filtering (dB)	SNR improvement (dB)
LMMN, LMF, LMMN, LMF	2.2116	14.9435	12.7319
LMS, LMS, LMS, LMS	2.2116	14.0935	11.8819
LMF, LMF, LMF, LMF	2.2116	14.8909	12.6793
LMMN, LMMN, LMMN, LMMN	2.2116	14.2994	12.0878
SRLMMN, SRLMF, SRLMMN, SRLMF	2.2116	13.6959	11.4843

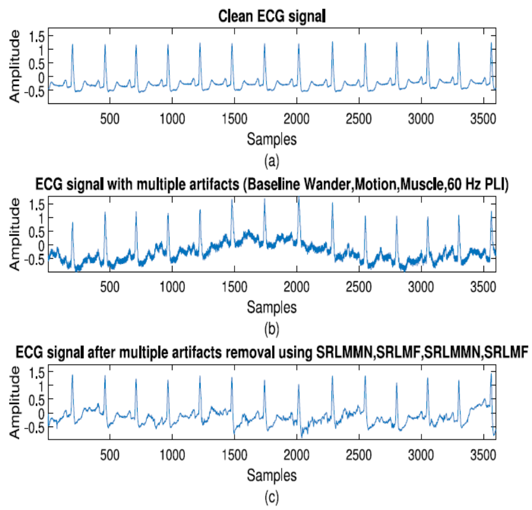

Fig. 9. (a) MIT-BIH Arrhythmia Database (MITDB) Record: 105, (b) MIT-BIH Arrhythmia Database (MITDB) Record: 105 + MIT-BIH Noise Stress Test Database (NSTDB) Record: bw + MIT-BIH Noise Stress Test Database (NSTDB) Record: em + MIT-BIH Noise Stress Test Database (NSTDB) Record: ma + 60 Hz PLI, (c) Recovered MIT-BIH Arrhythmia Database (MITDB) Record: 105 using the proposed cascaded 4-stage adaptive noise canceller employing the SRLMMN, SRLMF, SRLMMN, SRLMF adaptive algorithms for multiple artifacts removal (worst-case scenario).

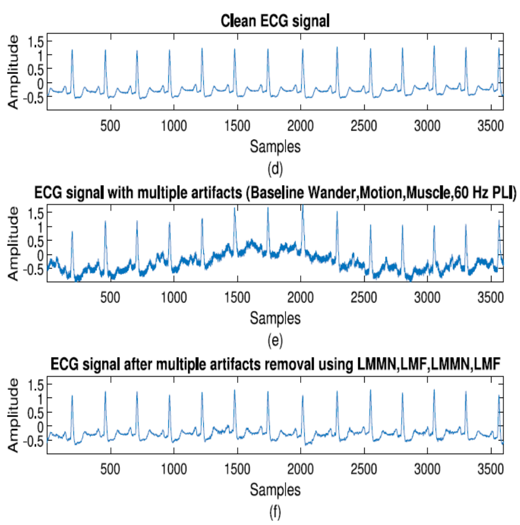

Fig. 10. (d) MIT-BIH Arrhythmia Database (MITDB) Record: 105, (e) MIT-BIH Arrhythmia Database (MITDB) Record: 105 + MIT-BIH Noise Stress Test Database (NSTDB) Record: bw + MIT-BIH Noise Stress Test Database (NSTDB) Record: em + MIT-BIH Noise Stress Test Database (NSTDB) Record: ma + 60 Hz PLI, (f) Recovered MIT-BIH Arrhythmia Database (MITDB) Record: 105 using the proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF adaptive algorithms for multiple artifacts removal (best-case scenario).

to remove the respective ECG artifacts as mentioned above. We succeeded in achieving an SNR improvement of 12.7319 dBs, which is better than the other compared methods. It is found that the proposed cascaded 4-stage adaptive noise canceller employing the LMMN, LMF, LMMN, LMF algorithms outperforms those that employ the LMS, LMS, LMS, LMS algorithms, the LMF, LMF, LMF, LMF algorithms, the LMMN, LMMN, LMMN, LMMN algorithms, and the SRLMMN, SRLMF, SRLMMN, SRLMF algorithms in terms of SNR improvement. It is also found that the performance of a single-stage adaptive noise canceller employing the SSLMS algorithm is comparatively poor in terms of SNR improvement as compared to the other seven algorithms studied in this work, viz. LMS, LMF, LMMN, SRLMS, SELMS, SRLMF, and SRLMMN. The different types of normalized adaptive algorithms and their respective sign counterparts in identifying the best candidates for the removal of multiple artifacts from the ECG signal using adaptive filters in cascade as discussed in this work will be the subject of our future studies.

CRediT authorship contribution statement

Mohammed Mujahid Ulla Faiz: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Writing – original draft, Writing – review & editing, Visualization. **Izzet Kale:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Writing – review & editing, Visualization, Supervision, Project administration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

$var\_noise = 0.1;$

$sqn = sqrt(var\_noise);$

$load('105m');$

$input = val(1,:)/200;$

$load('bwm');$

$v = sqn * val(1,:)/200;$

$snr\_before = snr(input, v);$

$snr\_after = snr(input, y);$

Appendix B

$load('105m');$

$input = val(1, :)/200;$

$f = 60;$

$fs = 360;$

$t = [1 : N]/fs;$

$v = 0.1 * sin(2 * pi * f * t + randn);$

$snr\_before = snr(input, v);$

$snr\_after = snr(input, y);$

Appendix C

$var\_noise = 0.1;$

$sqn = sqrt(var\_noise);$

$load('105m');$

$input = val(1, :)/200;$

$load('bwm');$

$v1 = sqn * val(1, :)/200;$

$load('emm');$

$v2 = sqn * val(1, :)/200;$

$load('mam');$

$v3 = sqn * val(1, :)/200;$

$f = 60;$

$fs = 360;$

$t = [1 : N]/fs;$

$v4 = 0.1 * sin(2 * pi * f * t + randn);$

$v = v1 + v2 + v3 + v4;$

$snr\_before = snr(input, v);$

$snr\_after = snr(input, y);$

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4 The NSRLMF Algorithm

4.1 Introduction

The Normalized Sign Regressor Least Mean Fourth (NSRLMF) algorithm is based on the clipping of the input data. The NSRLMF algorithm belongs to the family of the Normalized Least Mean Fourth (NLMF) algorithm [60]. The difference in the filter weights update equations of these two algorithms is the application of the signum function on the input data of the NSRLMF algorithm and the manner in which normalization has been applied.

The filter weights update equation of the NSRLMF algorithm for real-valued data is given by (4.1) [35], [36]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2} \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \tag{4.1}$$

where $\mathbf{w}_i$ is the updated filter weight vector at iteration $i \geq 0$, μ is the step-size, ϵ is a small positive constant to avoid division by zero when the regressor is zero, $\mathbf{u}_i$ is the regressor vector, $e_i = d_i - y_i$ is the estimation error signal, d_i is the desired signal, y_i is the adaptive filter output, $\mathrm{sign}(.)$ denotes the sign of its argument, the definitions of the signum function for real-valued data is given by (1.1), $||\mathbf{u}_i||_{\mathrm{H}}^2 = \mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, and $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined by:

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i1}|}, \frac{1}{|\mathbf{u}_{i2}|}, \dots, \frac{1}{|\mathbf{u}_{iM}|}\right\}, \tag{4.2}$$

where M is the filter length and $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$.

4.2 Background

The Normalized Sign Regressor Least Mean Square (NSRLMS) algorithm, which is the counterpart of the NSRLMF algorithm has been studied extensively in the open literature. However, there were no efforts made to study the performance evaluation of the NSRLMF algorithm until it was proposed, analyzed, and evaluated in [35], [36].

The normalization term present in the NSRLMF algorithm has been introduced in order to enhance its convergence performance compared to the Sign Regressor Least Mean Fourth (SRLMF) algorithm. The motivation to introduce the sign regressor term in the NSRLMF algorithm is to achieve reduced computational complexity compared to the NLMF algorithm. However, the convergence performance of the NSRLMF algorithm is slower than the NLMF algorithm but better than the NSRLMS algorithm.

In [56], [57], the NSRLMF algorithm is successfully employed by other researchers for power quality improvement in wind-solar based distributed generation system under harmonically distorted grid. The NSRLMF algorithm is shown to outperform the Least Mean Fourth (LMF) algorithm by providing enhanced dynamic response amidst sudden system variations [56], [57]. It should be noted that the authors in [56] published their expanded work in [57] at the time of making minor amendments to my thesis.

4.3 Contributions/Published Manuscripts

The two published papers on the performance evaluation of the NSRLMF [35], [36] algorithm for real-valued data are as follows:

[P6] M. M. U. Faiz and A. Zerguine, "The ε-Normalized Sign Regressor Least Mean Fourth (NSRLMF) adaptive algorithm," ***in Proc. of the 11th IEEE Int. Conf. on Information Sciences, Signal Processing and their Applications (ISSPA 2012),*** **Montreal, QC, Canada, pp. 339–342, July 2012, DOI: https://doi.org/10.1109/ISSPA.2012.6310571**

A novel adaptive algorithm called the NSRLMF algorithm was proposed, analyzed, and evaluated for the case of real-valued data in [35]. The expression for the steady-state Mean Square Error (MSE) $\varphi = \mathrm{E}[{e_i}^2]$ of the NSRLMF algorithm was derived and is given by (4.3) [35]:

$$\varphi = \frac{\mu\phi_1\xi_v^6}{6\sigma_v^2\phi_2 - 15\mu\phi_1\xi_v^4} + \sigma_v^2, \tag{4.3}$$

where $\phi_1 = \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{\left(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2\right)^2}\right]$, (4.4)

$$\phi_2 = \mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right], \tag{4.5}$$

wherein $\sigma_v^2 = \mathrm{E}[{v_i}^2]$ is the noise variance, and $\xi_v^4 = \mathrm{E}[{v_i}^4]$ and $\xi_v^6 = \mathrm{E}[{v_i}^6]$ are the fourth and sixth-order moments of the noise sequence v_i, respectively. Moreover, it is shown that the simulation results are in a good match with the analytical results for both white Gaussian and correlated Gaussian regressors.

Finally, a comparison between the convergence performance of the NSRLMF and NLMF algorithms indicates a slight performance degradation of the NSRLMF algorithm for white Gaussian regressors in a uniform noise environment with an SNR of 10 dB.

[P7] M. M. U. Faiz and A. Zerguine, "Convergence and tracking analysis of the ε-NSRLMF algorithm," ***in Proc. of the 38th IEEE Int. Conf. on Acoustics, Speech, and Signal Processing (ICASSP 2013),*** **Vancouver, BC, Canada, pp. 5657–5660, May 2013, DOI: https://doi.org/10.1109/ICASSP.2013.6638747**

The convergence and tracking behaviors of the NSRLMF algorithm were analyzed and evaluated for the case of real-valued data in [36]. The expression for the tracking MSE φ' of the NSRLMF algorithm was derived and is given by (4.6) [36]:

$$\varphi' = \frac{\mu\phi_1\xi_v^6 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{6\sigma_v^2\phi_2 - 15\mu\phi_1\xi_v^4} + \sigma_v^2, \tag{4.6}$$

where $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$ of the noise sequence $\mathbf{q}_i$. In addition, the expression for the optimum step-size μ_{opt} of the NSRLMF algorithm was also derived and is given by (4.7) [36]:

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\xi_v^6}\left[\frac{25\left(\xi_v^4\right)^2\mathrm{Tr}(\mathbf{Q})}{4\left(\sigma_v^2\right)^2(\phi_2)^2\xi_v^6} + \frac{1}{\phi_1}\right]} - \frac{5\xi_v^4\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2\phi_2\xi_v^6}. \tag{4.7}$$

Furthermore, the stability bound on the step-size of the NSRLMF algorithm to ensure convergence in the mean was also derived and is given by (4.8) [36]:

$$0 < \mu < \frac{2}{1+3\sigma_v^2}. \tag{4.8}$$

It is clear from (4.8) that the upper bound on the step-size of the NSRLMF algorithm no longer depends on the maximum eigenvalue $\lambda_{\max}$ of the regressor covariance matrix $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i]$ as was in the case of the SRLMF algorithm [32]. Moreover, it is shown that the simulation results are in a good match with the analytical results for both white Gaussian and correlated Gaussian regressors.

Finally, a comparison between the convergence performance of the NSRLMF and NLMF algorithms indicates that the effect of clipping on the performance of the NSRLMF algorithm is more evident for correlated Gaussian data than white Gaussian data in both Additive White Gaussian Noise (AWGN) and uniform noise environments with an SNR of 10 dB. This results in slower convergence of the NSRLMF algorithm for correlated Gaussian data than white Gaussian data compared to the NLMF algorithm.

THE $\epsilon-$NORMALIZED SIGN REGRESSOR LEAST MEAN FOURTH (NSRLMF) ADAPTIVE ALGORITHM

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ABSTRACT

In this paper, a new algorithm, the $\epsilon-$normalized sign regressor least mean fourth (NSRLMF) algorithm is presented as a substitute for the $\epsilon-$normalized least mean fourth (NLMF) algorithm. This new algorithm reduces significantly the computational load. Moreover, the proposed algorithm has similar convergence properties as those of the $\epsilon-$NLMF algorithm. Finally, simulations corroborate very well the theoretical findings.

1. INTRODUCTION

The sign based variants of the least mean square (LMS) algorithm [1] were introduced in order to reduce its computational and implementation costs [2]–[3]. The sign regressor algorithm (SRA) is one such variant of the LMS algorithm, which is based on clipping of the input data [4]. However, these sign based algorithms result in slower convergence speeds when compared with the LMS algorithm [5]–[6].

In [7], it is shown that the normalized least mean fourth (NLMF) algorithm exhibits faster convergence than the least mean fourth (LMF) algorithm [8]. Convergence and steady-state analysis of the NLMF algorithm are found in [9].

The above mentioned advantages motivates us to analyze and design the proposed $\epsilon-$normalized sign regressor least mean fourth (NSRLMF) algorithm, which is the normalized version of the sign regressor least mean fourth (SRLMF) algorithm [10]. In this paper, the mean-square analysis of the $\epsilon-$NSRLMF algorithm is developed. The framework used in our analysis relies on energy conservation arguments [11]. From the simulation results it is shown that the theoretical and simulated results are in good agreement. Moreover, the results show that the $\epsilon-$NSRLMF algorithm exhibits faster convergence than the LMF and SRLMF algorithms and slightly slower convergence than the $\epsilon-$NLMF algorithm for the same steady-state mean-square error (MSE).

The rest of the paper is organized as follows. Section 2 deals with a more explicit development of the proposed algorithm, and Section 3 treats its mean-square analysis. The computational load of the proposed algorithm is detailed in Section 4, while the performance evaluation of the resulting algorithm is carried out in Section 5. Finally, the conclusion section summarizes this work.

2. ALGORITHM DEVELOPMENT

Consider a zero-mean random variable d with realizations $\{d(0), d(1), \ldots\}$, and a zero-mean random row vector $\mathbf{u}$ (regressor) with realizations $\{\mathbf{u}_0, \mathbf{u}_1, \ldots\}$, the optimal weight vector $\mathbf{w}^o$ that solves:

$$\min_{\mathbf{w}} \mathrm{E}|d - \mathbf{u}\mathbf{w}|^4, \tag{1}$$

can be approximated iteratively via the update equation (the $\epsilon-$NSRLMF algorithm)

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}} \,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \qquad i \geq 0, \tag{2}$$

where $\mathbf{w}_i$ (column vector) is the updated weight vector at time i, μ is the step-size, ϵ is a small positive constant to avoid division by zero when the regressor is zero, $||\mathbf{u}_i||^2_{\mathrm{H}} = \mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined by

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i_M}|}\right\}, \tag{3}$$

where M is the filter length, $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, and e_i denotes the estimation error given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}. \tag{4}$$

3. MEAN-SQUARE ANALYSIS OF THE $\epsilon-$NSRLMF ALGORITHM

We shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the stationary data model [11]:

A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that $d_i = \mathbf{u}_i \mathbf{w}^o + v_i$.

A.2 The noise sequence v_i is independent and identically distributed (i.i.d.) with variance $\sigma_v^2 = \mathrm{E}[v_i^2]$ and is independent of $\mathbf{u}_j$ for all i, j.

A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.

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A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^T \mathbf{u}_i] > \mathbf{0}$.

For the adaptive filter of the form in (2), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [11]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] = 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \to \infty, \tag{5}$$

where

$$\begin{aligned} \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] &= \mathrm{E}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^T], & (6)\\ e_i &= e_{a_i} + v_i, & (7) \end{aligned}$$

with $\mathrm{g}[e_i]$ denoting some function of e_i, and $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error. Then $\mathrm{g}[e_i]$ for the $\epsilon-$NSRLMF algorithm becomes

$$\begin{aligned} \mathrm{g}[e_i] &= \frac{e_i^3}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2},\\ &= \frac{1}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\{e_{a_i}^3 + e_{a_i}v_i^2 + 2e_{a_i}^2 v_i + v_i e_{a_i}^2\\ &\quad + v_i^3 + 2e_{a_i}v_i^2\}. \qquad (8) \end{aligned}$$

By using the fact that e_{a_i} and v_i are independent, we reach at the following expression for the term $\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right]$:

$$\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right] = \mathrm{E}\left[\frac{e_{a_i}^4}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right] + 3\sigma_v^2 \mathrm{E}\left[\frac{e_{a_i}^2}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{9}$$

Ignoring third and higher-order terms of e_{a_i}, we obtain

$$\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right] \approx 3\sigma_v^2 \mathrm{E}\left[\frac{e_{a_i}^2}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{10}$$

To evaluate the term $\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right]$, we start by noting that

$$\begin{aligned} \mathrm{g}^2[e_i] &= \frac{1}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\Big[e_{a_i}^6 + 6e_{a_i}^5 v_i + 6e_{a_i}v_i^5\\ &+ 15e_{a_i}^4 v_i^2 + 15e_{a_i}^2 v_i^4 + 20e_{a_i}^3 v_i^3 + v_i^6\Big] \qquad (11) \end{aligned}$$

If we multiply $\mathrm{g}^2[e_i]$ by $||\mathbf{u}_i||_{\mathrm{H}}^2$ from the left, use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, and again ignoring third and higher-order terms of e_{a_i}, we obtain

$$\begin{aligned} \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] &\approx 6\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2 e_{a_i}}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\mathrm{E}[v_i^5]\\ +15\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2 e_{a_i}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\xi_v^4 &+ \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\xi_v^6, \qquad (12) \end{aligned}$$

where $\xi_v^4 = \mathrm{E}[v_i^4]$ and $\xi_v^6 = \mathrm{E}[v_i^6]$ denote the fourth and sixth-order moments of v_i, respectively.
Substituting (10) and (12) into (5) we get

$$\begin{aligned} 6\mu\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2 e_{a_i}}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\mathrm{E}[v_i^5] + 15\mu\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2 e_{a_i}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\xi_v^4\\ +\mu\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\xi_v^6 = 6\sigma_v^2\mathrm{E}\left[\frac{e_{a_i}^2}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \qquad (13) \end{aligned}$$

In order to simplify (13), we use the separation principle, namely, that at steady-state, $||\mathbf{u}_i||_{\mathrm{H}}^2$ is independent of e_{a_i} and $e_{a_i}^2$. Also, the first term in (13) will be zero since e_{a_i} is zero mean. Therefore, we obtain

$$\begin{aligned} 15\mu\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\mathrm{E}[e_{a_i}^2]\xi_v^4 + \mu\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\xi_v^6\\ = 6\sigma_v^2\mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\mathrm{E}[e_{a_i}^2]. \qquad (14) \end{aligned}$$

Now, let us define the following quantities:

$$\begin{aligned} \mathcal{Z}_1 &\triangleq \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right], & (15)\\ \mathcal{Z}_2 &\triangleq \mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right], & (16) \end{aligned}$$

then we can write (14) more compactly as

$$\begin{aligned} 15\mu\mathcal{Z}_1\mathrm{E}[e_{a_i}^2]\xi_v^4 + \mu\mathcal{Z}_1\xi_v^6 = 6\sigma_v^2\mathcal{Z}_2\mathrm{E}[e_{a_i}^2],\\ \mu\mathcal{Z}_1\xi_v^6 = (6\sigma_v^2\mathcal{Z}_2 - 15\mu\mathcal{Z}_1\xi_v^4)\mathrm{E}[e_{a_i}^2]. \qquad (17) \end{aligned}$$

Therefore, the expression for the steady-state excess-mean-square error (EMSE) $\zeta = \mathrm{E}[e_{a_i}^2]$ of the $\epsilon-$NSRLMF algorithm is given by

$$\zeta = \frac{\mu\mathcal{Z}_1\xi_v^6}{(6\sigma_v^2\mathcal{Z}_2 - 15\mu\mathcal{Z}_1\xi_v^4)}. \tag{18}$$

When ϵ is sufficiently small, which is usually the case, then its effect can be ignored. Therefore,

$$\mathcal{Z}_1 = \mathcal{Z}_2 = \mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{19}$$

In this case, expression (18) becomes

$$\zeta = \frac{\mu\xi_v^6}{(6\sigma_v^2 - 15\mu\xi_v^4)}, \tag{20}$$

which is independent of the regressor.

An alternative expression for the steady-state EMSE of the $\epsilon-$NSRLMF algorithm can be obtained by using the assumption $\epsilon \approx 0$ in order to simplify (13) into

$$\begin{aligned} 6\mu\mathrm{E}\left[\frac{e_{a_i}}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\mathrm{E}[v_i^5] + 15\mu\mathrm{E}\left[\frac{e_{a_i}^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\xi_v^4\\ +\mu\mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\xi_v^6 = 6\sigma_v^2\mathrm{E}\left[\frac{e_{a_i}^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \qquad (21) \end{aligned}$$

Now, let us use the following steady-state approximations:

$$\begin{aligned} \mathrm{E}\left[\frac{e_{a_i}}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] &\approx \frac{\mathrm{E}[e_{a_i}]}{\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]}, & (22)\\ \mathrm{E}\left[\frac{e_{a_i}^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] &\approx \frac{\mathrm{E}[e_{a_i}^2]}{\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]}. & (23) \end{aligned}$$

From [10], we have

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}). \tag{24}$$

Substituting (22), (23) and (24) into (21) we get

$$15\mu\sqrt{\frac{\pi\sigma_u^2}{2}}\frac{\mathrm{E}[e_{a_i}^2]}{\mathrm{Tr}(\mathbf{R})}\xi_v^4+\mu\mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\xi_v^6$$
$$=6\sigma_v^2\sqrt{\frac{\pi\sigma_u^2}{2}}\frac{\mathrm{E}[e_{a_i}^2]}{\mathrm{Tr}(\mathbf{R})}. \qquad (25)$$

Therefore, the steady-state EMSE of the $\epsilon-$NSRLMF algorithm can also be approximated by

$$\zeta=\frac{\mu\mathrm{Tr}(\mathbf{R})}{(6\sigma_v^2-15\mu\xi_v^4)}\sqrt{\frac{2}{\pi\sigma_u^2}}\xi_v^6\mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \qquad (26)$$

4. COMPUTATIONAL LOAD

In this section, the computational load of the proposed algorithm is compared with other algorithms in the family. Tables 1 and 2 report this comparison for real- and complex-valued data, respectively, in terms of the number of real additions (+), real multiplications (×), real divisions (/), and sign evaluations per iteration. As can be seen from Table 1, the real-valued data case, both the $\epsilon-$NLMF and $\epsilon-$NSRLMF algorithms have equal computational complexity, while there are $2M$ extra additions and multiplications for the $\epsilon-$NLMF algorithm when compared to the $\epsilon-$NSRLMF algorithm in the complex-valued data case as reported in Table 2.

Table 1. Computational load for real-valued data.

Algorithm	$+$	$\times$	$/$	sign
LMF	$2M$	$2M+3$		
SRLMF	$2M$	$2M+2$		1
$\epsilon-$NLMF	$3M$	$3M+3$	1	
$\epsilon-$NSRLMF	$3M$	$3M+2$	1	1

Table 2. Computational load for complex-valued data.

Algorithm	$+$	$\times$	$/$	sign
LMF	$8M+1$	$8M+5$		
SRLMF	$6M+1$	$6M+3$		2
$\epsilon-$NLMF	$10M+1$	$10M+5$	1	
$\epsilon-$NSRLMF	$8M+1$	$8M+3$	2	2

5. SIMULATION RESULTS

To assess the performance of our proposed algorithm, extensive simulations are carried out for this purpose. First, the theoretical findings are tested. Figures 1-2 illustrate the steady-state MSE of a 10-tap filter using white and correlated Gaussian regressors, respectively, for the $\epsilon-$NSRLMF algorithm. Here, the MSE is plotted versus the step-size μ for a signal to noise ratio (SNR) of 30 dB and the value of ϵ is set to 10^{-6}. As can be seen from these figures, the simulation results match very well the theoretical ones ((20) and (26)), which are the first and second approximations of the steady-state EMSE of the proposed algorithm, respectively. Similarly, Figure 3 shows the MSE behavior of the $\epsilon-$NSRLMF algorithm when tested using Gaussian regressors with an eigenvalue spread of five. As can be seen from this figure, the simulation results are found to reasonably corroborate with the second approximation of the steady-state EMSE of the proposed algorithm in particular.

Second, the performance of our proposed $\epsilon-$NSRLMF algorithm is compared with those of the LMF, SRLMF, and $\epsilon-$NLMF algorithms in an unknown system identification scenario with

$$\mathbf{w}^o=[0.227\ \ 0.460\ \ 0.688\ \ 0.460\ \ 0.227]^{\mathrm{T}}. \qquad (27)$$

Figure 4 depicts the convergence behavior of all four algorithms using white Gaussian regressors in a uniform noise environment for an SNR of 10 dB. One can notice from this figure that the $\epsilon-$NSRLMF algorithm converges faster than the LMF and SRLMF algorithms, while slightly slower than the $\epsilon-$NLMF algorithm. Similar behavior, as depicted in Figure 5, can be observed for the third-tap weight of all the algorithms.

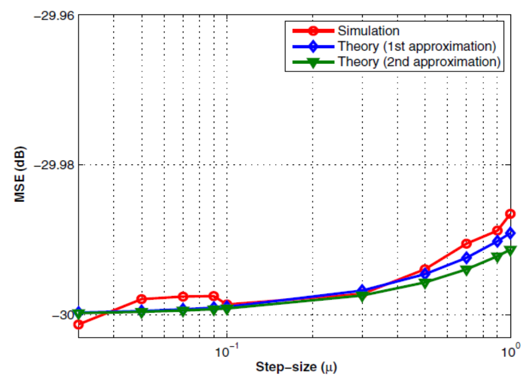

Fig. 1. Steady-state MSE of the $\epsilon-$NSRLMF algorithm using white Gaussian regressors.

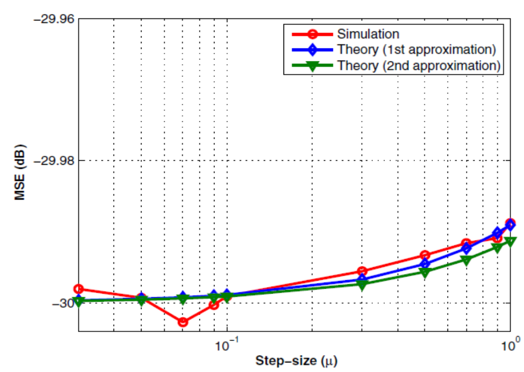

Fig. 2. Steady-state MSE of the $\epsilon-$NSRLMF algorithm using correlated Gaussian regressors.

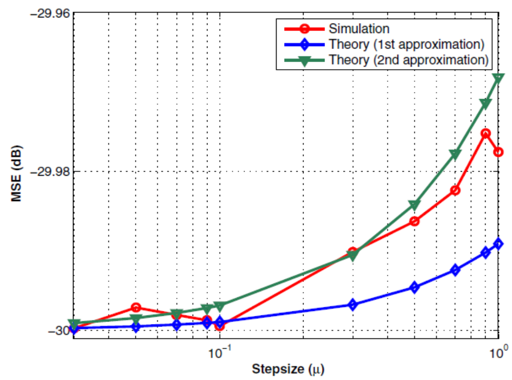

Fig. 3. Steady-state MSE of the $\epsilon-$NSRLMF algorithm using Gaussian regressors with an eigenvalue spread=5.

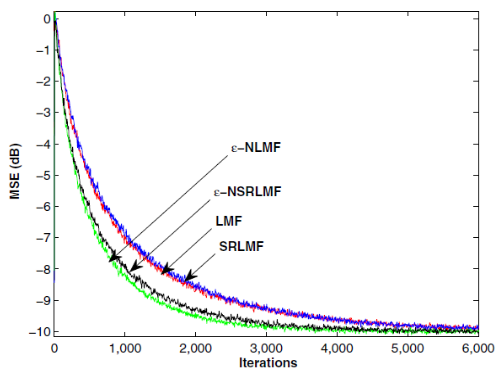

Fig. 4. Learning curves for LMF, SRLMF, $\epsilon-$NLMF, and $\epsilon-$NSRLMF algorithms in a uniform noise environment.

6. CONCLUSIONS

Closed-form analytical expressions are derived for the steady-state EMSE behavior of the $\epsilon-$NSRLMF algorithm. In addition, the computational complexity of the proposed algorithm is compared with those of LMF, SRLMF, and $\epsilon-$NLMF algorithms. Simulations performed are found to corroborate with the analytical results. Finally, it was shown that the $\epsilon-$NSRLMF algorithm has a slight performance loss when compared with the $\epsilon-$NLMF algorithm.

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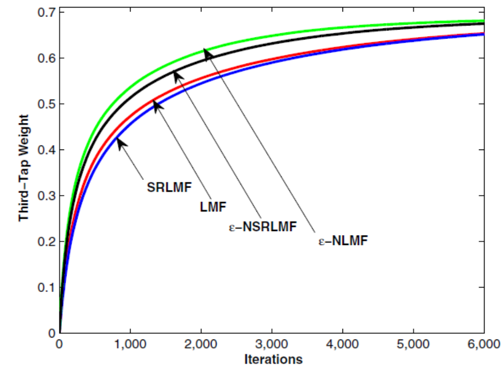

Fig. 5. Third-tap weight of LMF, SRLMF, $\epsilon-$NLMF, and $\epsilon-$NSRLMF algorithms in a uniform noise environment.

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CONVERGENCE AND TRACKING ANALYSIS OF THE $\epsilon-$NSRLMF ALGORITHM

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ABSTRACT

In this work, the convergence and tracking behavior of the $\epsilon-$normalized sign regressor least mean fourth (NSRLMF) algorithm are analyzed in the presence of white and correlated Gaussian data. Furthermore, the stability bound on the step-size of the $\epsilon-$NSRLMF algorithm to ensure convergence in the mean, which also leads us to the mean convergence of the $\epsilon-$normalized sign regressor least mean mixed-norm (NSRLMMN) algorithm is derived. Finally, simulation results are conducted to confirm the validity and performance of the proposed adaptive algorithm for both white and correlated Gaussian regressors.

Index Terms— LMF, NLMF, SRLMF, NSRLMF, Convergence, Tracking.

1. INTRODUCTION

The normalized least mean fourth (NLMF) algorithm was introduced for two reasons [1]–[2]. First, to get better convergence rate as compared to the traditional least mean fourth (LMF) algorithm [3]. Second, to overcome the convergence dependency of the LMF algorithm on the input data correlation statistics.

On the other hand, the sign regressor least mean fourth (SRLMF) algorithm, which is based on clipping of the input data, was introduced in order to reduce the complexity of the LMF algorithm [4]. Then, it was also observed that the LMF and SRLMF algorithms converge at an almost identical rate for the case of real-valued data. However, the convergence behavior of both of these algorithms depends on the input data correlation statistics [3]–[4].

Motivated by the advantages of sign adaptive filters and NLMF algorithm as mentioned above we introduced the normalized version of the SRLMF algorithm and performed its steady-state analysis in [5]. In the present paper, the convergence and tracking behavior of the $\epsilon-$NSRLMF algorithm is analyzed and very well supported by simulations.

The remainder of the paper is organized as follows. In Section 2, a brief description of the $\epsilon-$NSRLMF algorithm is provided, while in Section 3, the tracking analysis of the $\epsilon-$NSRLMF algorithm is derived. Section 4 deals with the convergence analysis of the proposed algorithm. Simulation results are reported in Section 5 to validate the theoretical findings. Finally, Section 6 concludes the paper.

2. THE $\epsilon-$NSRLMF ALGORITHM

The weight update recursion of the $\epsilon-$NSRLMF algorithm is given by the following expression:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \qquad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ is the updated weight vector, μ is the step-size, $\mathbf{u}_i$ is the regressor vector, ϵ is a small positive constant to avoid division by zero when the regressor is zero, e_i is the estimation error, $||\mathbf{u}_i||^2_{\mathrm{H}} = \mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined by

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i\mathrm{M}}|}\right\}, \tag{2}$$

M is the filter length and $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$.

3. TRACKING ANALYSIS

Tracking analysis of the $\epsilon-$NSRLMF algorithm can be extended in a straightforward way using its mean-square analysis presented in [5] as there are only slight differences. We will therefore be brief in this section.

Here, let us assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the nonstationary data model [6]:

A.1 There exists an optimal weight vector $\mathbf{w}_i^o$ such that $d_i = \mathbf{u}_i \mathbf{w}_i^o + v_i$, where d_i is the desired sequence and v_i is the noise sequence with variance σ_v^2.

A.2 The weight vector varies according to the random-walk model $\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i$, and the sequence $\mathbf{q}_i$ is independent and identically distributed (i.i.d.) with covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.

A.3 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [6]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}} \mathrm{g}^2[e_i]\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) \quad = \quad 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \to \infty, \tag{3}$$

where $\mathrm{g}[e_i]$ denotes some function of e_i and for the $\epsilon-$NSRLMF algorithm $\mathrm{g}[e_i]$ is readily given by

$$\begin{aligned} \mathrm{g}[e_i] &= \frac{e_i^3}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}, \\ &= \frac{1}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\{e^3_{a_i} + v_i^3 + 3e^2_{a_i}v_i + 3e_{a_i}v_i^2\}, \end{aligned} \tag{4}$$

$e_{a_i} = \mathbf{u}_i(\mathbf{w}_i^o - \mathbf{w}_{i-1})$ is the a priori estimation error, the estimation error is

$$e_i \quad = \quad e_{a_i} + v_i, \tag{5}$$

and finally

$$\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}] \quad = \quad \mathrm{E}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}]. \tag{6}$$

In [5], the following expressions for the terms $\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right]$ and $\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\mathrm{g}^2[e_i]\right]$ were derived:

$$\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right] = \mathrm{E}\left[\frac{e^4_{a_i}}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\right] + 3\sigma_v^2 \mathrm{E}\left[\frac{e^2_{a_i}}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\right], \tag{7}$$

$$\begin{aligned} \mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\mathrm{g}^2[e_i]\right] &\approx 6\mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}} e_{a_i}}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\mathrm{E}[v_i^5] \\ &+ 15\mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}} e^2_{a_i}}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\xi_v^4 + \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\xi_v^6, \end{aligned} \tag{8}$$

where $\xi_v^4 = \mathrm{E}[v_i^4]$ and $\xi_v^6 = \mathrm{E}[v_i^6]$ denote the fourth- and sixth-order moments of v_i, respectively. Finally, substituting expressions (7) and (8) into (3) we get

$$\mu \mathcal{Z}_1 \xi_v^6 + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = (6\sigma_v^2 \mathcal{Z}_2 - 15\mu \mathcal{Z}_1 \xi_v^4)\mathrm{E}[e^2_{a_i}], \tag{9}$$

where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are defined, respectively, as

$$\mathcal{Z}_1 \quad \triangleq \quad \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right], \tag{10}$$

$$\mathcal{Z}_2 \quad \triangleq \quad \mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\right]. \tag{11}$$

Therefore, the expression for the tracking excess-mean-square error (EMSE) $\zeta = \mathrm{E}[e^2_{a_i}]$ for the $\epsilon-$NSRLMF algorithm is given by

$$\zeta = \frac{\mu \mathcal{Z}_1 \xi_v^6 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{(6\sigma_v^2 \mathcal{Z}_2 - 15\mu \mathcal{Z}_1 \xi_v^4)}. \tag{12}$$

Consequently, the optimum step-size of the $\epsilon-$NSRLMF algorithm can be obtained by minimizing (12) with respect to μ and can be shown to be

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\xi_v^6}\left[\frac{25(\xi_v^4)^2 \mathrm{Tr}(\mathbf{Q})}{4\sigma_v^4(\mathcal{Z}_2)^2\xi_v^6} + \frac{1}{\mathcal{Z}_1}\right]} - \frac{5\xi_v^4 \mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2 \mathcal{Z}_2 \xi_v^6}. \tag{13}$$

Finally, the corresponding minimum value of the tracking mean-square error (MSE) of the $\epsilon-$NSRLMF algorithm is derived derived straight forward from (12) and is given by

$$\mathrm{E}\left[e_i^2\right] = \frac{\mu_{\mathrm{opt}} \mathcal{Z}_1 \xi_v^6 + \mu_{\mathrm{opt}}^{-1}\mathrm{Tr}(\mathbf{Q})}{(6\sigma_v^2 \mathcal{Z}_2 - 15\mu_{\mathrm{opt}} \mathcal{Z}_1 \xi_v^4)} + \sigma_v^2. \tag{14}$$

4. CONVERGENCE ANALYSIS

To carry out the convergence analysis of the $\epsilon-$NSRLMF algorithm we rely on the following assumptions [2]:

A.4 The noise sequence v_i is independent of $\mathbf{u}_j$ for all i, j and both sequences have zero mean.

A.5 The weight error vector $\widetilde{\mathbf{w}}_i$ (defined below) is independent of the input $\mathbf{u}_j$ for all i, j.

Subtracting both sides of (1) from $\mathbf{w}_i^o$ we get

$$\widetilde{\mathbf{w}}_i = \widetilde{\mathbf{w}}_{i-1} - \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \tag{15}$$

where the weight error vector $\widetilde{\mathbf{w}}_i$ is given by

$$\widetilde{\mathbf{w}}_i = \mathbf{w}_i^o - \mathbf{w}_i. \tag{16}$$

We know that, the desired sequence d_i is given by

$$d_i = \mathbf{u}_i \mathbf{w}_i^o + v_i, \tag{17}$$

and the estimation error e_i is given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}. \tag{18}$$

Then substituting (16) and (17) into (18) and expanding the term e_i^3, we get

$$e_i^3 = (\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^3 + v_i^3 + 3(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^2 v_i + 3\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1}v_i^2. \tag{19}$$

At convergence [2], the following holds:

$$(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^3 \leq \mathbf{u}_i\widetilde{\mathbf{w}}_{i-1}. \tag{20}$$

Since $(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^3$ is a convex function for $\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1} \geq 0$, the above inequality is always true as long as $\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1} \leq 1$. Therefore, (19) can be approximated by

$$e_i^3 \approx \mathbf{u}_i\widetilde{\mathbf{w}}_{i-1} + v_i^3 + 3(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^2 v_i + 3\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1}v_i^2. \tag{21}$$

Substituting (21) into (15) we get

$$\begin{aligned}\widetilde{\mathbf{w}}_i &= \widetilde{\mathbf{w}}_{i-1} - \frac{\mu}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}\Big[\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1} + v_i^3 \\ &\quad +3(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^2 v_i + 3\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1}v_i^2\Big], \\ &= \left[\mathbf{I} - \frac{\mu}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\,\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}(1+3v_i^2)\right]\widetilde{\mathbf{w}}_{i-1} \\ &\quad - \frac{\mu\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}[v_i^3 + 3(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^2 v_i]. \end{aligned} \tag{22}$$

Taking the expectation of both sides of (22) under the above-mentioned assumptions and by ignoring ϵ as it is very small, we obtain

$$\mathrm{E}[\widetilde{\mathbf{w}}_i] = \left[\mathbf{I} - \mu\mathrm{E}\left[\frac{\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right](1+3\sigma_v^2)\right]\mathrm{E}[\widetilde{\mathbf{w}}_{i-1}]. \tag{23}$$

Now, let us use the following approximation:

$$\mathrm{E}\left[\frac{\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] \approx \frac{\mathrm{E}\left[\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}\right]}{\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]}. \tag{24}$$

From [4], we have

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \mathrm{E}\left[\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}\right] = \sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}). \tag{25}$$

Upon substituting (24) and (25) into (23), we have

$$\mathrm{E}[\widetilde{\mathbf{w}}_i] = \left[1 - \mu(1+3\sigma_v^2)\right]\mathrm{E}[\widetilde{\mathbf{w}}_{i-1}]. \tag{26}$$

From (26), it is easy to show that the mean behavior of the weight error vector, that is $\mathrm{E}[\widetilde{\mathbf{w}}_i]$, converges to the zero vector if the step-size μ is bounded by:

$$0 < \mu < \frac{2}{1+3\sigma_v^2}. \tag{27}$$

Note that the step-size bound of the $\epsilon-$NSRLMF algorithm in (27) is the same as that obtained for the NLMF algorithm in [2]. It is clear from (27) that the upper bound on the step-size of the $\epsilon-$NSRLMF algorithm no longer depends on the maximum eigenvalue, $\lambda_{\max}$, of the input data autocorrelation matrix as was in the case for the SRLMF algorithm [4].

In [7], it was mentioned that the step-size bound of the $\epsilon-$NSRLMMN algorithm can be obtained by combining the step-size bounds of the $\epsilon-$NSRLMS and $\epsilon-$NSRLMF algorithms. This is very clear from the fact that the $\epsilon-$NSRLMMN algorithm reduces to $\epsilon-$NSRLMF and $\epsilon-$NSRLMS algorithms when the mixing parameter, δ, takes the value 0 and 1, respectively. Therefore, by utilizing equation (27), the mean convergence of the $\epsilon-$NSRLMMN algorithm can now be approximated by:

$$0 < \mu_{\epsilon-\mathrm{NSRLMMN}} < 2\delta + \frac{2(1-\delta)}{1+3\sigma_v^2}. \tag{28}$$

5. SIMULATION RESULTS

Several simulation results are conducted to corroborate the theoretical findings in an unknown system identification scenario. For this purpose, $\epsilon = 10^{-6}$ and $\mathrm{M} = 5$ have fixed throughout this study. In Figures 1-2, the variance of the Gaussian noise sequence $\mathbf{q}_i$ in the random-walk model is fixed at $\sigma_q^2 = 10^{-8}$. Moreover, the correlated data can be obtained in the same way as was done in [7].

Figures 1-2 depict the tracking MSE of the $\epsilon-$NSRLMF algorithm using correlated and white Gaussian regressors, respectively. In these figures, the MSE is depicted as a function of the step-size for a signal-to-noise ratio (SNR) of 20 dB under an additive white Gaussian noise (AWGN) environment. It is seen in Figure 1 that the simulation results are in a close match with the analytical results for values of μ up to 0.5. A zoom into the region around $\mu = 0.1$ in Figure 1 shows that the tracking MSE possesses a minimum value of 0.01006151 at $\mu = 0.114$, which are in excellent agreement with the corresponding theoretical values of 0.01005815 and $\mu_{\mathrm{opt}} = 0.1143$ obtained from expressions (14) and (13), respectively. However, the simulation and analytical results are found to be in reasonable agreement for white Gaussian data as depicted in Figure 2.

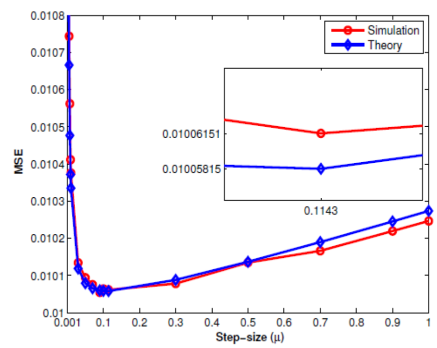

Fig. 1. Theoretical and simulated tracking MSE of the $\epsilon-$NSRLMF algorithm using correlated Gaussian regressors.

Finally, the results in Figures 3-4 compare the convergence behavior of the $\epsilon-$NSRLMF and $\epsilon-$NLMF algorithms in AWGN and uniform noise environments, respectively. In these figures, the convergence curves are plotted for both correlated and white Gaussian data at an SNR of 10 dB. As can be seen from these figures, the $\epsilon-$NLMF algorithm outperforms the $\epsilon-$NSRLMF algorithm with correlated Gaussian input. However, the performance of both algorithms is found to be similar in white Gaussian data.

6. CONCLUSIONS

In this work, expressions are derived for the tracking MSE and optimum step-size of the $\epsilon-$NSRLMF algorithm. A

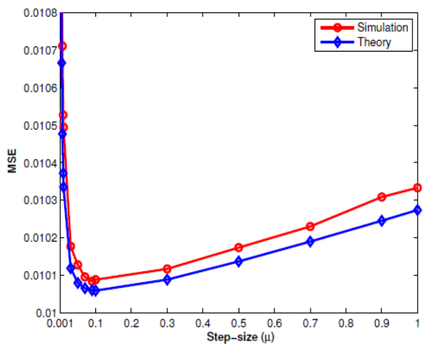

Fig. 2. Theoretical and simulated tracking MSE of the $\epsilon-$NSRLMF algorithm using white Gaussian regressors.

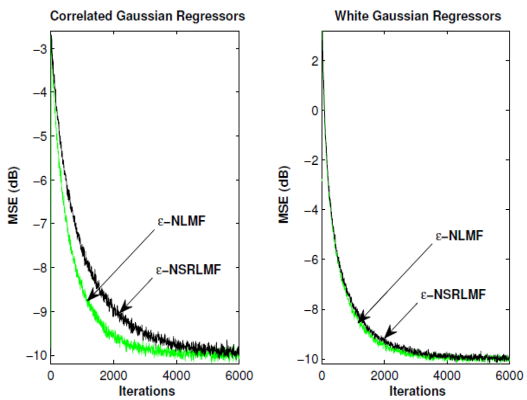

Fig. 3. Comparison of the MSE learning curves of $\epsilon-$NLMF and $\epsilon-$NSRLMF algorithms in AWGN environment.

sufficient condition for the convergence in the mean of the $\epsilon-$NSRLMF algorithm is also derived and is found to be the same as that of the NLMF algorithm. It is also shown that the upper bound on the step-size of the $\epsilon-$NSRLMF algorithm depends on the noise variance only and is independent of the input data correlation statistics. As a by-product of this work, the mean convergence of the $\epsilon-$NSRLMMN algorithm is also obtained. Finally, a close match between analytical and simulation results for correlated Gaussian data than white Gaussian data is obtained. Moreover, the effect of clipping on the performance of the $\epsilon-$NSRLMF algorithm is found to be more evident for correlated Gaussian data than white Gaussian data.

Current work is devised for the recently newly version of the NLMF algorithm [8]. Similarly, as was done in this work, future work is extending the presented idea to that in [8] and eventually compare their results. Due to the nature of the normalization in [8], it is expected that the analytical approach will be very involved.

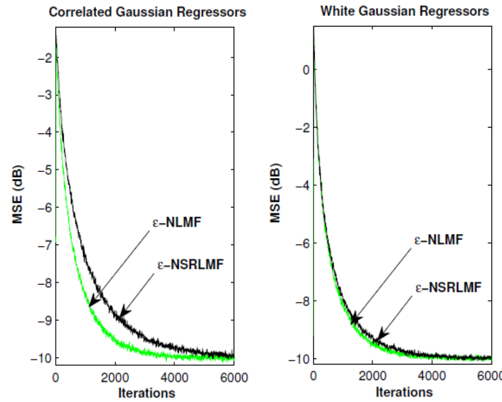

Fig. 4. Comparison of the MSE learning curves of $\epsilon-$NLMF and $\epsilon-$NSRLMF algorithms in a uniform noise environment.

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5 The NSRLMMN Algorithm

5.1 Introduction

The Normalized Sign Regressor Least Mean Mixed-Norm (NSRLMMN) algorithm is based on the clipping of the input data. The NSRLMMN algorithm belongs to the family of the Normalized Least Mean Mixed-Norm (NLMMN) algorithm. The difference in the filter weights update equations of these two algorithms is the application of the signum function on the input data of the NSRLMMN algorithm and the manner in which normalization has been applied. The NSRLMMN algorithm is a hybrid version of the Normalized Sign Regressor Least Mean Square (NSRLMS) and Normalized Sign Regressor Least Mean Fourth (NSRLMF) algorithms. The NSRLMMN algorithm combines the benefits of both the NSRLMS and NSRLMF algorithms such as improved stability and convergence performance, respectively.

The filter weights update equation of the NSRLMMN algorithm for real-valued data is given by (5.1) [37], [38]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}} \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i[\delta + (1-\delta)e_i^2], \tag{5.1}$$

where $\mathbf{w}_i$ is the updated filter weight vector at iteration $i \geq 0$, μ is the step-size, ϵ is a small positive constant to avoid division by zero when the regressor is zero, $\mathbf{u}_i$ is the regressor vector, $e_i = d_i - y_i$ is the estimation error signal, d_i is the desired signal, y_i is the adaptive filter output, δ is the mixing parameter ranging between $0 \leq \delta \leq 1$, $\mathrm{sign}(.)$ denotes the sign of its argument, the definitions of the signum function for real-valued data is given by (1.1), $||\mathbf{u}_i||^2_{\mathrm{H}} = \mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, and $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined by:

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i1}|}, \frac{1}{|\mathbf{u}_{i2}|}, \dots, \frac{1}{|\mathbf{u}_{iM}|}\right\}, \tag{5.2}$$

where M is the filter length and $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$. The filter weights update equation of the NSRLMMN algorithm reduces to the filter weights update equations of the NSRLMF and NSRLMS algorithms when the mixing parameter δ becomes 0 and 1, respectively.

5.2 Background

The NLMMN algorithm, which is the counterpart of the NSRLMMN algorithm has hardly received any attention in the open literature. Also, there were no efforts made to study the performance evaluation of the NSRLMMN algorithm until it was proposed, analyzed, and evaluated in [37], [38].

The normalization term present in the NSRLMMN algorithm has been introduced in order to enhance its convergence performance as compared to the Sign Regressor Least Mean Mixed-

Norm (SRLMMN) algorithm. The motivation to introduce the sign regressor term in the NSRLMMN algorithm is to achieve reduced computational complexity compared to the NLMMN algorithm. However, the convergence performance of the NSRLMMN algorithm is slower than the NSRLMF algorithm but better than the NSRLMS algorithm as expected.

5.3 Contributions/Published Manuscripts

The two published papers on the performance evaluation of the NSRLMMN [37], [38] algorithm for real-valued data are as follows:

[P8] M. M. U. Faiz and A. Zerguine, "Convergence analysis of the ε NSRLMMN algorithm," ***in Proc. of the 20th European Signal Processing Conf. (EUSIPCO 2012),*** **Bucharest, Romania, pp. 235–239, Aug. 2012, ISBN: 978-1-4673-1068-0**

A novel adaptive algorithm called the NSRLMMN algorithm was proposed, analyzed, and evaluated for the case of real-valued data in [37]. The expression for the steady-state Mean Square Error (MSE) $\varphi = \mathrm{E}[{e_i}^2]$ of the NSRLMMN algorithm was derived and is given by (5.3) [37]:

$$\varphi = \frac{\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)\phi_1}{2(\delta+3\bar{\delta}\sigma_v^2)\phi_2-\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)\phi_1} + \sigma_v^2, \tag{5.3}$$

where $\phi_1 = \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{\left(\epsilon+||\mathbf{u}_i||_{\mathrm{H}}^2\right)^2}\right]$, (5.4)

$$\phi_2 = \mathrm{E}\left[\frac{1}{\epsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right], \tag{5.5}$$

wherein $\bar{\delta} = 1 - \delta$, $\sigma_v^2 = \mathrm{E}[{v_i}^2]$ is the noise variance, and $\xi_v^4 = \mathrm{E}[{v_i}^4]$ and $\xi_v^6 = \mathrm{E}[{v_i}^6]$ are the fourth and sixth-order moments of the noise sequence v_i, respectively. We can obtain the expressions for the steady-state MSE of the NSRLMF and NSRLMS algorithms from (5.3) by setting δ equal to 0 and 1, respectively, as shown in [43].

In addition, a sufficient condition for the convergence in the mean of the NSRLMMN algorithm was also derived and is given by (5.6) [37]:

$$0 < \mu < 2\delta + \frac{2\bar{\delta}}{1+3\sigma_v^2}. \tag{5.6}$$

Similarly, we can obtain the expressions for the step-size bounds of the NSRLMF and NSRLMS algorithms from (5.6) by setting δ equal to 0 and 1, respectively, as shown in [37]. Moreover, it is shown that the simulation results are in a good match with the analytical results for both white Gaussian and correlated Gaussian regressors.

Finally, a comparison between the convergence performance of the NSRLMMN and NLMMN algorithms indicates performance degradation of the NSRLMMN algorithm in a uniform noise environment with an SNR of 10 dB.

[P9] M. M. U. Faiz and A. Zerguine, "Tracking analysis of the ε-NSRLMMN algorithm," ***in the Conf. Record of the 46th Asilomar Conf. on Signals, Systems, and Computers (Asilomar 2012),*** **Pacific Grove, CA, USA, pp. 816–819, Nov. 2012, DOI: https://doi.org/10.1109/ACSSC.2012.6489127**

The tracking behavior of the NSRLMMN algorithm was analyzed and evaluated for the case of real-valued data in [38]. The expression for the tracking MSE φ' of the NSRLMMN algorithm was derived and is given by (5.7) [38]:

$$\varphi' = \frac{\mu c\phi_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2a\phi_2 - \mu b\phi_1} + \sigma_v^2, \quad (5.7)$$

where $a = \delta + 3\bar{\delta}\sigma_v^2,$ (5.8)

$$b = \delta^2 + 15\bar{\delta}^2\xi_v^4 + 12\delta\bar{\delta}\sigma_v^2, \quad (5.9)$$

$$c = \delta^2\sigma_v^2 + \bar{\delta}^2\xi_v^6 + 2\delta\bar{\delta}\xi_v^4, \quad (5.10)$$

wherein $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$ of the noise sequence $\mathbf{q}_i$. We can obtain the expressions for the tracking MSE of the NSRLMF and NSRLMS algorithms from (5.7) by setting δ equal to 0 and 1, respectively, as shown in [43].

In addition, the expression for the optimum step-size μ_{opt} of the NSRLMMN algorithm was also derived and is given by (5.11) [38]:

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{c\phi_1}\left[1 + \frac{b^2\phi_1\mathrm{Tr}(\mathbf{Q})}{4a^2c{\phi_2}^2}\right]} - \frac{b\mathrm{Tr}(\mathbf{Q})}{2ac\phi_2}. \quad (5.11)$$

Similarly, we can obtain the expressions for the optimum step-size of the NSRLMF and NSRLMS algorithms from (5.11) by setting δ equal to 0 and 1, respectively, as shown in [43]. Moreover, it is shown that the simulation results are in a close match with the analytical results for correlated Gaussian regressors in particular.

Finally, a comparison between the convergence performance of the NSRLMMN and Least Mean Mixed-Norm (LMMN) algorithms indicates faster convergence of the NSRLMMN algorithm in a uniform noise environment with an SNR of 10 dB.

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CONVERGENCE ANALYSIS OF THE ε NSRLMMN ALGORITHM

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ABSTRACT

In this work, the $\varepsilon-$normalized sign regressor least mean mixed-norm (NSRLMMN) adaptive algorithm is proposed. The proposed algorithm exhibits increased convergence rate as compared to the least mean mixed-norm (LMMN) and the sign regressor least mean mixed-norm (SRLMMN) algorithms. Also, the steady-state analysis and convergence analysis are presented. Moreover, the proposed $\varepsilon-$NSRLMMN algorithm substantially reduces the computational load, a major drawback of the $\varepsilon-$normalized least mean mixed-norm (NLMMN) algorithm. Finally, simulation results are presented to support the theoretical findings.

Keywords: Adaptive filters, LMS, LMF, Least Mean Mixed-Norm (LMMN), Sign regressor LMMN algorithm.

1. INTRODUCTION

While the least mean mixed-norm (LMMN) algorithm was introduced in order to combine the advantages of both the least mean square (LMS) and the least mean fourth (LMF) algorithms [1]– [6], the sign adaptive filters were proposed in order to reduce the computational cost and to simplify the hardware implementation [7]– [8]. However, these sign adaptive filters result in slower convergence speeds due to clipping of the estimation error or the input data, or both [9]. The algorithm based on clipping of the input data of the LMMN is known as the sign regressor least mean mixed-norm (SRLMMN) algorithm. The convergence speed of the SRLMMN algorithm can be increased by normalizing it. Hence the name the $\varepsilon-$normalized sign regressor least mean mixed-norm (NSRLMMN) algorithm. From the simulation results it is shown that the $\varepsilon-$NSRLMMN algorithm outperforms both the LMMN and SRLMMN algorithms.

The paper is organized as follows. In Section 2, the $\varepsilon-$NSRLMMN algorithm is proposed. The steady-state analysis of the proposed algorithm is derived in Section 3, and Section 4 presents its convergence analysis. A comparison of the computational complexity of the proposed algorithm with those of other algorithms in the family is presented in Section 5. Finally, the simulation results and conclusions are presented in Sections 6 and 7, respectively.

2. THE $\varepsilon-$NSRLMMN ALGORITHM

Consider a zero-mean random variable d with realizations $\{d(0), d(1), \ldots\}$, and a zero-mean random row vector $\mathbf{u}$ with realizations $\{\mathbf{u}_0, \mathbf{u}_1, \ldots\}$. The LMMN algorithm is based on the following convex cost function [1]– [3]:

$$J_i = \mathsf{E}\left[\delta e_i^2 + (1-\delta)e_i^4\right], \qquad 0 \le \delta \le 1, \tag{1}$$

where δ is the mixing parameter and e_i denotes the estimation error given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}. \tag{2}$$

The update equation for the $\varepsilon-$NSRLMMN algorithm can be shown to be governed by the following recursion:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\varepsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2} \,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i [\delta + (1-\delta)e_i^2], \quad i \ge 0, \tag{3}$$

where $\mathbf{w}_i$ (column vector) is the updated weight vector at time i with optimal weight vector $\mathbf{w}^o$, μ is the step-size, ε is a small positive constant used for regularization purposes, $||\mathbf{u}_i||_{\mathrm{H}}^2 = \mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, and $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined by

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i_M}|}\right\}, \tag{4}$$

where M is the filter length and $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$.

3. STEADY-STATE ANALYSIS OF THE $\varepsilon-$NSRLMMN ALGORITHM

We shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the stationary data model [10]:

A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that $d_i = \mathbf{u}_i \mathbf{w}^o + v_i$.

A.2 The noise sequence v_i is independent and identically distributed (i.i.d.) with variance $\sigma_v^2 = \mathsf{E}[v_i^2]$ and is independent of $\mathbf{u}_j$ for all i, j.

A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.

A.4 The regressor covariance matrix is $\mathbf{R} = \mathsf{E}[\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_i] > \mathbf{0}$.

For the adaptive filter of the form in (3), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [10]:

$$\mu \mathsf{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] = 2\mathsf{E}\left[e_{a_i} \mathrm{g}[e_i]\right], \text{ as } i \to \infty, \tag{5}$$

where

$$\mathsf{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \mathsf{E}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}], \tag{6}$$

$$e_i = e_{a_i} + v_i, \tag{7}$$

with $\mathrm{g}[e_i]$ denoting some function of e_i, and $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error. Then $\mathrm{g}[e_i]$ for the

$\varepsilon-$NSRLMMN algorithm becomes

$$\begin{aligned} g[e_i] &= \frac{e_i[\delta+(1-\delta)e_i^2]}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}, \\ &= \frac{\delta(e_{a_i}+v_i)}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2} + \frac{\bar{\delta}}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\{e_{a_i}^3+e_{a_i}v_i^2 \\ &\quad +2e_{a_i}^2v_i+v_ie_{a_i}^2+v_i^3+2e_{a_i}v_i^2\}, \end{aligned} \tag{8}$$

where $\bar{\delta}=1-\delta$. By using the fact that e_{a_i} and v_i are independent, we reach at the following expression for the term $\mathsf{E}[e_{a_i}g[e_i]]$:

$$\begin{aligned} \mathsf{E}[e_{a_i}g[e_i]] &= \bar{\delta}\mathsf{E}\left[\frac{e_{a_i}^4}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right] + (\delta+3\bar{\delta}\sigma_v^2) \\ &\quad \times\mathsf{E}\left[\frac{e_{a_i}^2}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \end{aligned} \tag{9}$$

Ignoring third and higher-order terms of e_{a_i}, we obtain

$$\mathsf{E}[e_{a_i}g[e_i]] \approx (\delta+3\bar{\delta}\sigma_v^2)\mathsf{E}\left[\frac{e_{a_i}^2}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{10}$$

To evaluate the term $\mathsf{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 g^2[e_i]\right]$, we start by noting that

$$\begin{aligned} g^2[e_i] &= \frac{\delta^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}[e_{a_i}^2+v_i^2+2e_{a_i}v_i] \\ &\quad + \frac{\bar{\delta}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\Big[e_{a_i}^6+6e_{a_i}^5v_i+6e_{a_i}v_i^5 \\ &\quad +15e_{a_i}^4v_i^2+15e_{a_i}^2v_i^4+20e_{a_i}^3v_i^3+v_i^6\Big] \\ &\quad + \frac{2\delta\bar{\delta}}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\Big[e_{a_i}^4+6e_{a_i}^2v_i^2+4e_{a_i}^3v_i \\ &\quad +4e_{a_i}v_i^3+v_i^4\Big]. \end{aligned} \tag{11}$$

If we multiply $g^2[e_i]$ by $||\mathbf{u}_i||_{\mathrm{H}}^2$ from the left, use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, and again ignoring third and higher-order terms of e_{a_i}, we obtain

$$\begin{aligned} \mathsf{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 g^2[e_i]\right] &\approx (\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2) \\ &\times\mathsf{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2 e_{a_i}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right] + (\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4) \\ &\times\mathsf{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right], \end{aligned} \tag{12}$$

where $\xi_v^4=\mathsf{E}[v_i^4]$, $\xi_v^6=\mathsf{E}[v_i^6]$ denote the fourth and sixth-order moments of v_i, respectively.
Substituting (10) and (12) into (5) we get

$$\begin{aligned} &\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)\mathsf{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2 e_{a_i}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right] \\ &+\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)\mathsf{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right] \\ &= 2(\delta+3\bar{\delta}\sigma_v^2)\mathsf{E}\left[\frac{e_{a_i}^2}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \end{aligned} \tag{13}$$

In order to simplify (13), we use the separation principle, namely, that at steady-state, $||\mathbf{u}_i||_{\mathrm{H}}^2$ is independent of $e_{a_i}^2$. Therefore, we obtain

$$\begin{aligned} &\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)\mathsf{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\mathsf{E}[e_{a_i}^2] \\ &+\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)\mathsf{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right] \\ &= 2(\delta+3\bar{\delta}\sigma_v^2)\mathsf{E}\left[\frac{1}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\mathsf{E}[e_{a_i}^2], \end{aligned} \tag{14}$$

which can be set up compactly as

$$\begin{aligned} &\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)\mathscr{Z}_1 = [2(\delta+3\bar{\delta}\sigma_v^2)\mathscr{Z}_2 \\ &\quad -\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)\mathscr{Z}_1]\mathsf{E}[e_{a_i}^2], \end{aligned} \tag{15}$$

where

$$\mathscr{Z}_1 \triangleq \mathsf{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right], \tag{16}$$

$$\mathscr{Z}_2 \triangleq \mathsf{E}\left[\frac{1}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{17}$$

Therefore, the expression for the steady-state excess-mean-square error (EMSE) $\zeta=\mathsf{E}[e_{a_i}^2]$ of the $\varepsilon-$NSRLMMN algorithm is given by

$$\zeta = \frac{\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)\mathscr{Z}_1}{[2(\delta+3\bar{\delta}\sigma_v^2)\mathscr{Z}_2-\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)\mathscr{Z}_1]}. \tag{18}$$

When ε is sufficiently small, which is usually the case, then its effect can be ignored. Therefore, (16) and (17) reduce to

$$\mathscr{Z}_1 = \mathscr{Z}_2 = \mathsf{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{19}$$

In this case, (18) becomes

$$\zeta = \frac{\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)}{[2(\delta+3\bar{\delta}\sigma_v^2)-\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)]}, \tag{20}$$

which is independent of the regressor.

An alternative expression for the steady-state EMSE of the $\varepsilon-$NSRLMMN algorithm can be obtained by using the assumption $\varepsilon\approx 0$ in order to simplify (13) into

$$\begin{aligned} &\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)\mathsf{E}\left[\frac{e_{a_i}^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]+\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6 \\ &+2\delta\bar{\delta}\xi_v^4)\mathsf{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] = 2(\delta+3\bar{\delta}\sigma_v^2)\mathsf{E}\left[\frac{e_{a_i}^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \end{aligned} \tag{21}$$

Now, let us use the following steady-state approximation:

$$\mathsf{E}\left[\frac{e_{a_i}^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] \approx \frac{\mathsf{E}[e_{a_i}^2]}{\mathsf{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]}. \tag{22}$$

In [11], we have shown that

$$\mathsf{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}). \tag{23}$$

Substituting (22) and (23) into (21) we get

$$\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)\mathsf{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathbf{H}}^2}\right]=\Big[2(\delta+3\bar{\delta}\sigma_v^2)$$
$$-\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)\Big]\sqrt{\frac{\pi\sigma_u^2}{2}}\frac{\mathsf{E}[e_{a_i}^2]}{\mathrm{Tr}(\mathbf{R})}. \quad (24)$$

Therefore, the steady-state EMSE of the $\varepsilon-$NSRLMMN algorithm can also be approximated by

$$\zeta = \frac{\mu\mathrm{Tr}(\mathbf{R})(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)}{[2(\delta+3\bar{\delta}\sigma_v^2)-\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)]} \times\sqrt{\frac{2}{\pi\sigma_u^2}}\mathsf{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathbf{H}}^2}\right]. \quad (25)$$

Ultimately, an expression for the mean-square error (MSE) of the $\varepsilon-$NSRLMMN algorithm is given by

$$\mathsf{E}\left[e_i^2\right]=\zeta+\sigma_v^2. \quad (26)$$

4. CONVERGENCE ANALYSIS OF THE $\varepsilon-$NSRLMMN ALGORITHM

In [1], the approximate bound on the step-size of the LMMN algorithm was obtained by simply combining the step-size bounds of LMS and LMF. Similarly, the approximate bound on the step-size of our proposed $\varepsilon-$NSRLMMN algorithm can be obtained by combining the step-size bounds of $\varepsilon-$NSRLMS and $\varepsilon-$NSRLMF.

It was shown in [12] that the convergence in the mean for the $\varepsilon-$NSRLMS algorithm is guaranteed by the stability condition for the $\varepsilon-$NLMS algorithm, namely,

$$0<\mu_{\varepsilon-\mathrm{NSRLMS}}<2. \quad (27)$$

Also, the mean convergence of the $\varepsilon-$NSRLMF algorithm can be bounded by

$$0<\mu_{\varepsilon-\mathrm{NSRLMF}}<\mu_{\mathrm{upper}}. \quad (28)$$

Thus, by combining (27) and (28) the mean convergence of the $\varepsilon-$NSRLMMN algorithm can be approximated by

$$0<\mu_{\varepsilon-\mathrm{NSRLMMN}}<2\delta+(1-\delta)\mu_{\mathrm{upper}}. \quad (29)$$

From (29) it is clear that the $\varepsilon-$NSRLMMN algorithm reduces to $\varepsilon-$NSRLMF and $\varepsilon-$NSRLMS algorithms when $\delta=0$ and $\delta=1$, respectively. Our future work will focus on finding the upper bound for the step-size of the $\varepsilon-$NSRLMF algorithm.

5. COMPUTATIONAL COMPLEXITY

In this section, the computational complexity of the $\varepsilon-$NSRLMMN algorithm is compared with those of other algorithms in the family, e.g., LMMN, SRLMMN, and $\varepsilon-$NLMMN algorithms. Tables 1 and 2 present this comparison for real- and complex-valued data, respectively, in terms of the number of real additions (+), real multiplications (×), real divisions (/), and comparisons with zero per iteration. Moreover, M is the filter order. As can be seen from Table 1, the real-valued data case, both the $\varepsilon-$NLMMN and $\varepsilon-$NSRLMMN algorithms have similar computational complexity, while there are $2M$ and $2M+4$ extra additions and multiplications per iteration, respectively, for the $\varepsilon-$NLMMN algorithm when compared to the $\varepsilon-$NSRLMMN algorithm in the complex-valued data case as reported from Table 2.

Table 1: Computational cost for real-valued data.

Algorithm	+	×	/	**sign**
LMMN	$2M+2$	$2M+4$		
SRLMMN	$2M+2$	$2M+2$		2
$\varepsilon-$NLMMN	$4M+2$	$4M+4$	2	
$\varepsilon-$NSRLMMN	$4M+2$	$4M+2$	2	2

Table 2: Computational cost for complex-valued data.

Algorithm	+	×	/	**sign**
LMMN	$8M+3$	$8M+6$		
SRLMMN	$6M+3$	$6M+2$		4
$\varepsilon-$NLMMN	$10M+3$	$10M+6$	2	
$\varepsilon-$NSRLMMN	$8M+3$	$8M+2$	4	4

6. SIMULATION RESULTS

In order to evaluate the steady-state and convergence performance of our proposed algorithm, extensive simulations are carried out for this purpose. The parameter settings in this study are as follows. In all the simulations, we have chosen $\varepsilon=10^{-6}$, the mixing parameter is fixed at $\delta=0.5$ (except Figures 5-6), and the filter length is fixed at $M=10$ for Figures 1-2 and $M=5$ for Figures 3-8.

First, the steady-state MSE of the $\varepsilon-$NSRLMMN algorithm using white and correlated Gaussian regressors is shown in Figures 1-2, respectively. In Figure 2, the correlated data is obtained by passing a unit-variance i.i.d. Gaussian data through a first-order auto-regressive model with transfer function $\frac{\sqrt{1-a^2}}{(1-az^{-1})}$ and $a=0.8$. In Figures 1-2, the MSE is plotted as a function of the step-size μ in additive white Gaussian noise (AWGN) environment for a signal to noise ratio (SNR) of 30 dB. As observed from these figures, the simulation results are in a good match with the theoretical results ((20) and (25)), which are, respectively, the first and second approximations of the steady-state EMSE of the $\varepsilon-$NSRLMMN algorithm. Also, as can be seen from these figures, the theoretical results are in a better match with the simulation results for correlated Gaussian data than white Gaussian data.

Second, the convergence behavior of the $\varepsilon-$NSRLMMN algorithm is compared with that of LMMN, SRLMMN, and $\varepsilon-$NLMMN algorithms in an unknown system identification setup with

$$\mathbf{w}^o=[0.227\ \ 0.460\ \ 0.688\ \ 0.460\ \ 0.227]^{\mathrm{T}}. \quad (30)$$

Figure 3 shows the convergence performance of all the four algorithms using white Gaussian regressors in a uniform noise environment with SNR = 10 dB. As it is depicted from this figure, the $\varepsilon-$NSRLMMN algorithm results in superior performance over the LMMN and SRLMMN algorithms, but is only slightly inferior when compared to the $\varepsilon-$NLMMN algorithm. Also, it is interesting to note that the performance of the SRLMMN algorithm is found to be identical to that of the LMMN algorithm for the same misadjustment. No deterioration has occurred to the SRLMMN algorithm. One can also observe this particular behavior from Figure 4, which shows the comparison of the third-tap weight learning curves of all the four algorithms for the same parameter settings.

Third, Figures 5-6 demonstrate, respectively, the MSE and normalized weight error vector learning behaviors of the $\varepsilon-$NSRLMMN algorithm for different values of the mixing parameter δ in a uniform noise environment at SNR $= 10$ dB. As can be seen from these figures, the $\varepsilon-$NSRLMMN algorithm boils down to $\varepsilon-$NSRLMF and $\varepsilon-$NSRLMS algorithms when $\delta = 0$ and $\delta = 1$, respectively. Therefore, by controlling δ we can control the tradeoff between fast convergence rate and small misadjustment. We also find that for uniform noise, the $\varepsilon-$NSRLMF algorithm is superior to both $\varepsilon-$NSRLMS and $\varepsilon-$NSRLMMN algorithms.

Finally, Figures 7-8 illustrate, respectively, the MSE and normalized weight error vector convergence behaviors of the $\varepsilon-$NSRLMMN algorithm in uniform, Gaussian and Laplacian noise environments for SNR $= 10$ dB. As can be seen from Figure 7 that the best performance in terms of convergence behavior is obtained with uniform noise while the worst performance is obtained with Laplacian noise. We also note from Figure 8 that the lowest weight error is reached by the proposed algorithm for uniform noise environment as compared to Gaussian and Laplacian noise environments.

7. CONCLUSIONS

In this work, the $\varepsilon-$NSRLMMN algorithm is presented and resulted in a significant reduction in computational load over the $\varepsilon-$NLMMN algorithm. The proposed $\varepsilon-$NSRLMMN algorithm has been shown to exhibit slightly slower convergence rate than the $\varepsilon-$NLMMN algorithm for the same steady-state error. The mean-square analysis of the $\varepsilon-$NSRLMMN algorithm is performed and is found to corroborate the simulation results. Also, the convergence behavior of the proposed algorithm is analyzed for different values of the mixing parameter and different noise environments.

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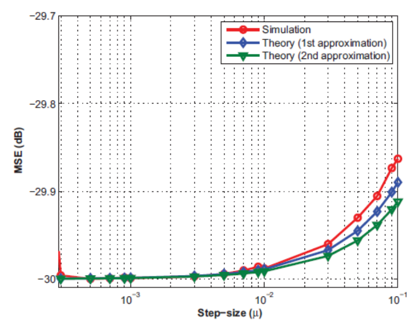

Figure 1: MSE performance of the $\varepsilon-$NSRLMMN algorithm using white Gaussian regressors in AWGN environment with SNR = 30 dB.

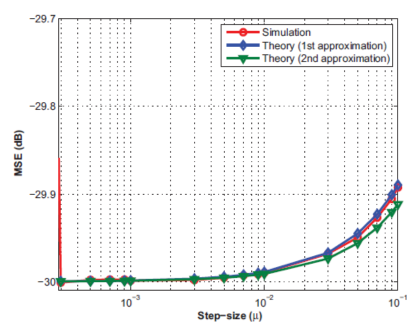

Figure 2: MSE performance of the $\varepsilon-$NSRLMMN algorithm using correlated Gaussian regressors in AWGN environment with SNR = 30 dB.

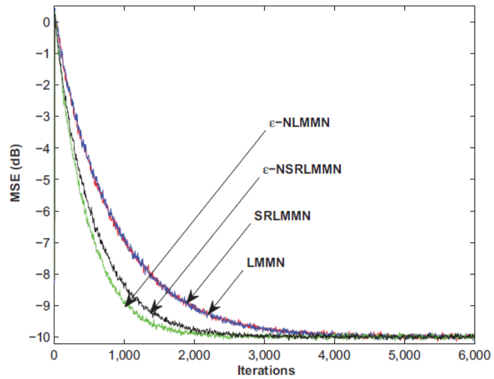

Figure 3: Comparison of the MSE learning curves of LMMN, SRLMMN, $\varepsilon-$NLMMN, and $\varepsilon-$NSRLMMN algorithms in a uniform noise environment with SNR = 10 dB.

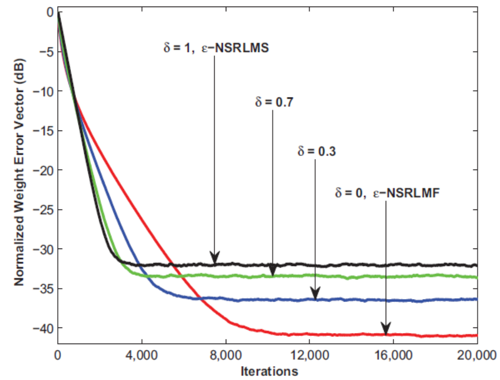

Figure 6: Comparison of the normalized weight error vector learning curves of the $\varepsilon-$NSRLMMN algorithm for different values of δ in a uniform noise environment with SNR = 10 dB.

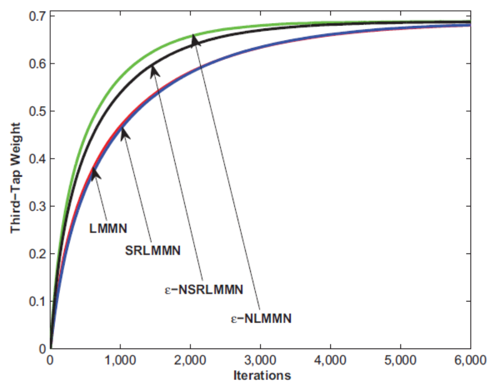

Figure 4: Comparison of the third-tap weight learning curves of LMMN, SRLMMN, $\varepsilon-$NLMMN, and $\varepsilon-$NSRLMMN algorithms in a uniform noise environment with SNR = 10 dB.

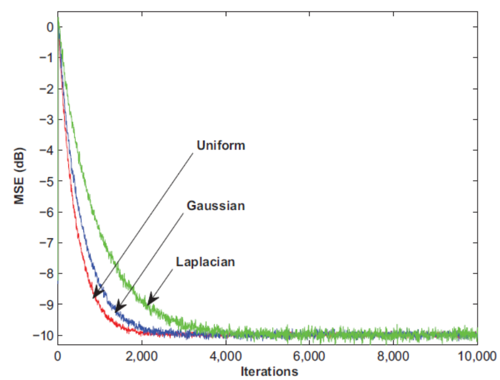

Figure 7: Comparison of the MSE learning curves of the $\varepsilon-$NSRLMMN algorithm in uniform, Gaussian and Laplacian noise environments with SNR = 10 dB.

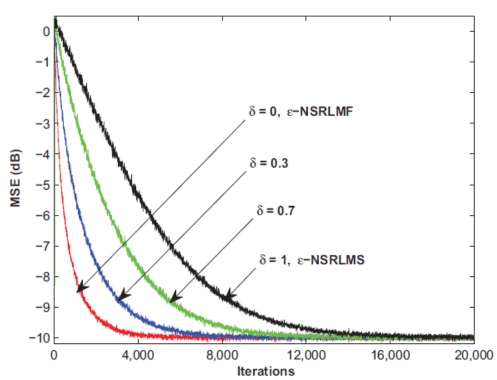

Figure 5: Comparison of the MSE learning curves of the $\varepsilon-$NSRLMMN algorithm for different values of δ in a uniform noise environment with SNR = 10 dB.

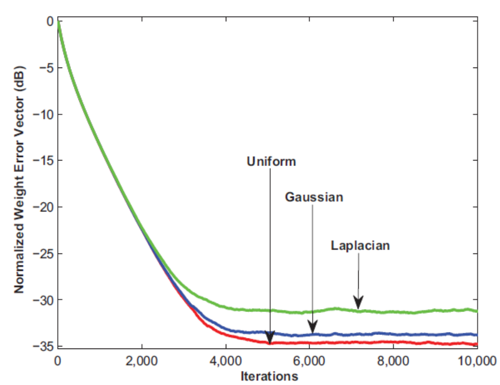

Figure 8: Comparison of the normalized weight error vector learning curves of the $\varepsilon-$NSRLMMN algorithm in uniform, Gaussian and Laplacian noise environments with SNR = 10 dB.

TRACKING ANALYSIS OF THE $\varepsilon-$NSRLMMN ALGORITHM

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ABSTRACT

In this work, expressions for the tracking excess-mean-square error (EMSE) and optimum step-size of the $\varepsilon-$normalized sign regressor least mean mixed-norm (NSRLMMN) adaptive algorithm are derived. Finally, extensive simulation results performed are found to corroborate very closely with the theoretical results for correlated Gaussian data.

1. INTRODUCTION

The least mean mixed-norm (LMMN) algorithm was introduced in order to combine the advantages of the least mean square (LMS) and the least mean fourth (LMF) algorithms, for example, the LMMN algorithm is known to have better steady-state performance than the LMS algorithm and better stability properties than the LMF algorithm [1]–[7].

In [8], a new approach to perform the tracking analysis of the LMF and LMMN algorithms was presented. This new approach bypassed the need for working directly with the weight error vector and was based on a fundamental energy-preserving relation [9].

In this work, we propose a simplified version of the LMMN algorithm called the $\varepsilon-$normalized sign regressor least mean mixed-norm (NSRLMMN) algorithm. This new algorithm makes use of the signum of the reference input signal, thereby reducing the computational cost and simplifying the hardware implementation [10]–[14]. The normalization term in the update recursion of the $\varepsilon-$NSRLMMN algorithm ensures that the proposed algorithm provides an increased convergence rate as compared to the LMMN algorithm.

To perform the tracking analysis of the $\varepsilon-$NSRLMMN algorithm we have used the modified energy-preserving relation presented in [9], in order to deal with the sign regressor term present in the update recursion of the $\varepsilon-$NSRLMMN algorithm.

The paper is structured as follows. In Section 2, the $\varepsilon-$NSRLMMN algorithm is introduced. The tracking analysis of the proposed algorithm is derived in Section 3. Finally, simulation results and conclusions are presented in Sections 4 and 5, respectively.

2. THE $\varepsilon-$NSRLMMN ALGORITHM

Consider a zero-mean random variable d with realizations $\{d_0, d_1, \ldots\}$, and a zero-mean random row vector $\mathbf{u}$ with realizations $\{\mathbf{u}_0, \mathbf{u}_1, \ldots\}$. The LMMN algorithm is based on the following convex cost function [3]–[4]:

$$J_i = \mathrm{E}\left[\delta e_i^2 + (1-\delta)e_i^4\right], \qquad 0 \leq \delta \leq 1, \tag{1}$$

where δ is the mixing parameter between the two error norms and e_i denotes the estimation error given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}. \tag{2}$$

The update recursion of the $\varepsilon-$NSRLMMN algorithm can be shown to be governed by the following recursion:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\varepsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2} \,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i [\delta + (1-\delta)e_i^2], \quad i \geq 0, \tag{3}$$

where $\mathbf{w}_i$ (column vector) is the updated weight vector at time i, μ is the step-size, ε is a small positive constant to avoid division by zero when the regressor is zero, $||\mathbf{u}_i||_{\mathrm{H}}^2 = \mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, and $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined by

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i_M}|}\right\}, \tag{4}$$

where M is the filter length and $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$.

3. TRACKING ANALYSIS

We shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the nonstationary data model [9]:

A.1 There exists a vector $\mathbf{w}_i^o$ such that $d_i = \mathbf{u}_i \mathbf{w}_i^o + v_i$, where v_i is the additive noise.

A.2 The weight vector varies according to the random-walk model $\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i$, and the sequence $\mathbf{q}_i$ is independent and identically distributed (i.i.d.) with covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.

A.3 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

For the adaptive filter of the form in (3), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [9]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \to \infty, \tag{5}$$

where

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \mathrm{E}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}], \tag{6}$$

$$e_i = e_{a_i} + v_i, \tag{7}$$

with $\mathrm{g}[e_i]$ denoting some function of e_i, $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error, and for the $\varepsilon-$NSRLMMN algorithm $g[e_i]$ can be set up into the following:

$$\begin{aligned} g[e_i] &= \frac{e_i[\delta+(1-\delta)e_i^2]}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}, \\ &= \frac{\delta(e_{a_i}+v_i)}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}+\frac{\bar{\delta}}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\{e_{a_i}^3+e_{a_i}v_i^2 \\ &\quad +2e_{a_i}^2v_i+v_ie_{a_i}^2+v_i^3+2e_{a_i}v_i^2\}, \end{aligned} \tag{8}$$

where $\bar{\delta}=1-\delta$. By using the fact that e_{a_i} and v_i are independent, we reach at the following expression for the term $\mathrm{E}[e_{a_i}g[e_i]]$:

$$\begin{aligned} \mathrm{E}[e_{a_i}g[e_i]] &= \bar{\delta}\mathrm{E}\left[\frac{e_{a_i}^4}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]+(\delta+3\bar{\delta}\sigma_v^2) \\ &\quad \times\mathrm{E}\left[\frac{e_{a_i}^2}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \end{aligned} \tag{9}$$

Ignoring third and higher-order terms of e_{a_i}, we obtain

$$\begin{aligned} \mathrm{E}[e_{a_i}g[e_i]] &\approx (\delta+3\bar{\delta}\sigma_v^2)\mathrm{E}\left[\frac{e_{a_i}^2}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right], \\ &\approx a\mathrm{E}\left[\frac{e_{a_i}^2}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right], \end{aligned} \tag{10}$$

where $a=\delta+3\bar{\delta}\sigma_v^2$.

To evaluate the term $\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2g^2[e_i]\right]$, we start by noting that

$$\begin{aligned} g^2[e_i] &= \frac{\delta^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}[e_{a_i}^2+v_i^2+2e_{a_i}v_i] \\ &\quad +\frac{\bar{\delta}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\Big[e_{a_i}^6+6e_{a_i}^5v_i+6e_{a_i}v_i^5 \\ &\quad +15e_{a_i}^4v_i^2+15e_{a_i}^2v_i^4+20e_{a_i}^3v_i^3+v_i^6\Big] \\ &\quad +\frac{2\delta\bar{\delta}}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\Big[e_{a_i}^4+6e_{a_i}^2v_i^2+4e_{a_i}^3v_i \\ &\quad +4e_{a_i}v_i^3+v_i^4\Big]. \end{aligned} \tag{11}$$

If we multiply $g^2[e_i]$ by $||\mathbf{u}_i||_{\mathrm{H}}^2$ from the left, use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, and again ignoring third and higher-order terms of e_{a_i}, we obtain

$$\begin{aligned} &\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2g^2[e_i]\right]\approx(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2) \\ &\times\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2e_{a_i}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]+(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4) \\ &\times\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right], \end{aligned} \tag{12}$$

where $\xi_v^4=\mathrm{E}[v_i^4]$ and $\xi_v^6=\mathrm{E}[v_i^6]$ denote the fourth and sixth-order moments of v_i, respectively. Let $b=\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2$ and $c=\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4$. Therefore, (12) looks like

$$\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2g^2[e_i]\right]\approx b\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2e_{a_i}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]+c\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]. \tag{13}$$

Substituting (10) and (13) into (5) we obtain

$$\begin{aligned} &\mu b\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2e_{a_i}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]+\mu c\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right] \\ &\quad +\mu^{-1}\mathrm{Tr}(\mathbf{Q})=2a\mathrm{E}\left[\frac{e_{a_i}^2}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \end{aligned} \tag{14}$$

In order to simplify (14), we use the separation principle, namely, that at steady-state, $||\mathbf{u}_i||_{\mathrm{H}}^2$ is independent of $e_{a_i}^2$. Ultimately, (14) becomes

$$\begin{aligned} &\mu b\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]\mathrm{E}[e_{a_i}^2]+\mu c\mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right] \\ &\quad +\mu^{-1}\mathrm{Tr}(\mathbf{Q})=2a\mathrm{E}\left[\frac{1}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\mathrm{E}[e_{a_i}^2]. \end{aligned} \tag{15}$$

Or more compactly as

$$\mu c\mathscr{Z}_1+\mu^{-1}\mathrm{Tr}(\mathbf{Q})=\left[2a\mathscr{Z}_2-\mu b\mathscr{Z}_1\right]\mathrm{E}[e_{a_i}^2], \tag{16}$$

where

$$\mathscr{Z}_1 \triangleq \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right], \tag{17}$$

and

$$\mathscr{Z}_2 \triangleq \mathrm{E}\left[\frac{1}{\varepsilon+||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{18}$$

Consequently, the expression for the steady-state excess-mean-square error (EMSE), $\zeta=\mathrm{E}[e_{a_i}^2]$, of the $\varepsilon-$NSRLMMN algorithm is given by

$$\zeta=\frac{\mu c\mathscr{Z}_1+\mu^{-1}\mathrm{Tr}(\mathbf{Q})}{[2a\mathscr{Z}_2-\mu b\mathscr{Z}_1]}. \tag{19}$$

An optimum value of the step-size of the $\varepsilon-$NSRLMMN algorithm can be obtained by minimizing (19) with respect to μ. Therefore, we get

$$\mu_{\mathrm{opt}}=\sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{c\mathscr{Z}_1}\left[1+\frac{b^2\mathscr{Z}_1\mathrm{Tr}(\mathbf{Q})}{4a^2c\mathscr{Z}_2^2}\right]}-\frac{b\mathrm{Tr}(\mathbf{Q})}{2ac\mathscr{Z}_2}, \tag{20}$$

and the corresponding minimum value of the steady-state EMSE of the $\varepsilon-$NSRLMMN algorithm is given by

$$\zeta_{\min}=\frac{\mu_{\mathrm{opt}}c\mathscr{Z}_1+\mu_{\mathrm{opt}}^{-1}\mathrm{Tr}(\mathbf{Q})}{[2a\mathscr{Z}_2-\mu_{\mathrm{opt}}b\mathscr{Z}_1]}. \tag{21}$$

4. SIMULATION RESULTS

To support the theoretical results, several simulations are carried out in order to assess the tracking performance of the $\varepsilon-$NSRLMMN algorithm. Throughout this study, unless otherwise stated, we have chosen $\varepsilon = 10^{-6}$, $\delta = 0.5$ (except Figures 5 and 6), and $M = 10$ under additive white Gaussian noise (AWGN) environment with a signal to noise ratio (SNR) of 20 dB.

Figures 1 and 2 compare the theoretical MSE obtained from expression (19) with the experimental MSE using correlated Gaussian regressors for smaller and larger values of μ, respectively. As can be seen from Fig. 1 that the theoretical and experimental MSE are in excellent match for smaller values of μ. Moreover, Fig. 1 shows that the steady-state MSE possesses a minimum value of 0.0100872 at $\mu = 0.017$, which are in very good agreement with the corresponding theoretical values of 0.0100897 and $\mu_{\text{opt}} = 0.0173$ obtained from expressions (21) and (20), respectively. Furthermore, the theoretical and experimental MSE are found to be in good match for larger values of μ as can be seen from Fig. 2.

Similarly, Fig. 3 and Fig. 4 present the same comparison for the case of white Gaussian regressors. As can be seen from these figures, the simulation results are found to be in reasonable agreement with the theoretical results.

Figures 5 and 6 compare the theoretical MSE obtained from expression (19) with the experimental MSE using correlated and white Gaussian regressors for different values of δ, respectively. From the figures, it can be seen that the minimum value of the MSE occurs at $\delta = 1$ for AWGN noise.

Finally, Fig. 7 depicts the convergence performance of the LMMN and the $\varepsilon-$NSRLMMN algorithms in an unknown system identification setup with

$$\mathbf{w}^{\bullet} = [0.227\ \ 0.460\ \ 0.688\ \ 0.460\ \ 0.227]^{\mathrm{T}}. \tag{22}$$

As it is depicted from this figure, the $\varepsilon-$NSRLMMN algorithm results in a superior performance over that of the LMMN algorithm.

5. CONCLUSIONS

In this work, the tracking analysis of the $\varepsilon-$NSRLMMN algorithm is carried out. In order to validate our theoretical findings simulations are performed for both cases of white and correlated Gaussian regressors. It is shown that the analytical results are in a close match with the simulation results for the correlated Gaussian regressor case in particular. Also, it is shown that the $\varepsilon-$NSRLMMN algorithm exhibits increased convergence rate as compared to that of the LMMN algorithm.

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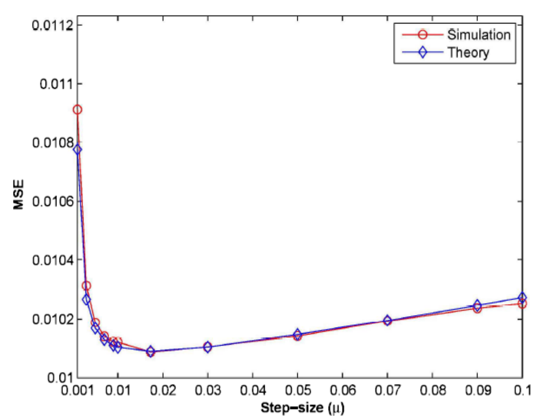

Figure 1: Theoretical and simulated MSE for smaller values of μ using correlated Gaussian regressors.

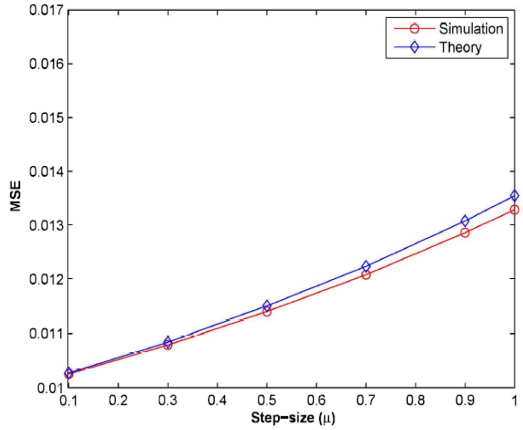

Figure 2: Theoretical and simulated MSE for larger values of μ using correlated Gaussian regressors.

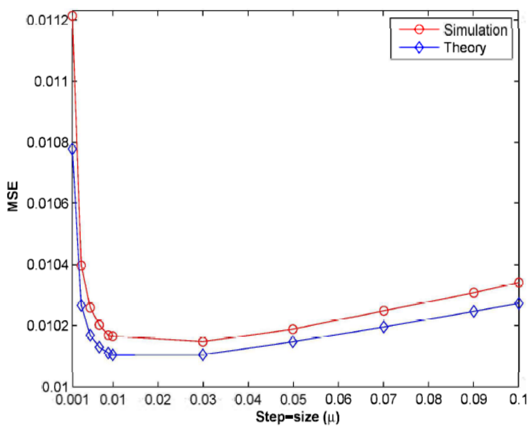

Figure 3: Theoretical and simulated MSE for smaller values of μ using white Gaussian regressors.

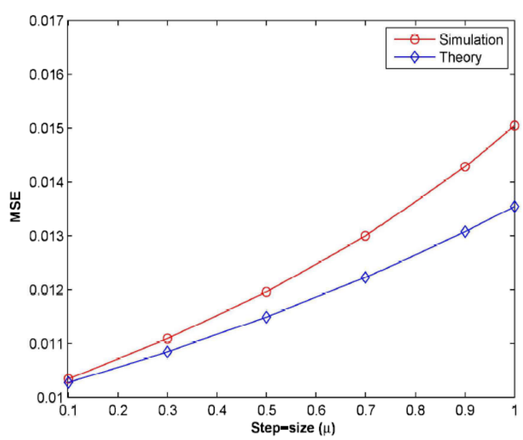

Figure 4: Theoretical and simulated MSE for larger values of μ using white Gaussian regressors.

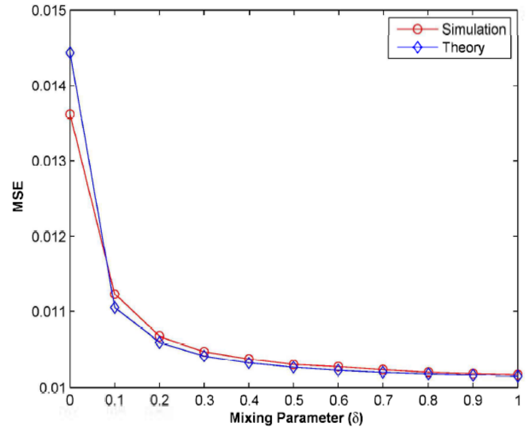

Figure 5: Theoretical and simulated MSE for different values of δ using correlated Gaussian regressors.

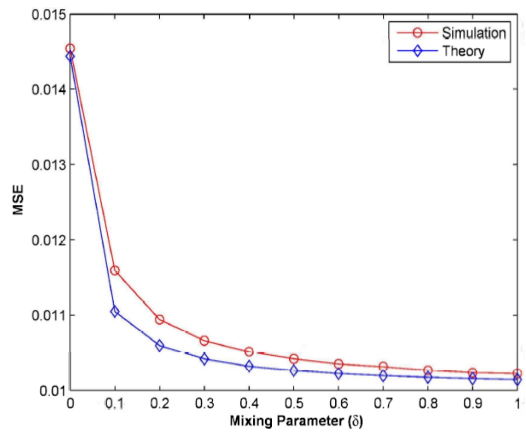

Figure 6: Theoretical and simulated MSE for different values of δ using white Gaussian regressors.

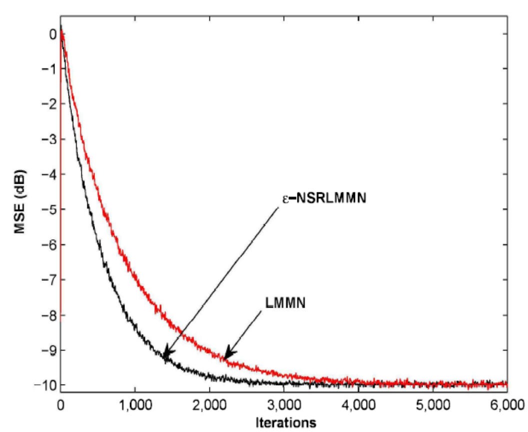

Figure 7: Comparison of the MSE learning curves of LMMN and $\varepsilon-$NSRLMMN algorithms using white Gaussian regressors in a uniform noise environment with SNR = 10 dB.

6 Other Sign Adaptive Algorithms

6.1 Introduction

6.1.1 The SRLMS Algorithm

The Sign Regressor Least Mean Square (SRLMS) algorithm is based on the clipping of the input data. The SRLMS algorithm belongs to the family of the Least Mean Square (LMS) algorithm. The only difference in the filter weights update equations of these two algorithms is the application of the signum function on the input data of the SRLMS algorithm.

The filter weights update equation of the SRLMS algorithm for complex-valued data is given by (6.1) [39]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \, \mathrm{csgn}[\mathbf{u}_i]^* e_i. \tag{6.1}$$

6.1.2 The SSLMS Algorithm

The Sign-Sign Least Mean Square (SSLMS) algorithm is based on the clipping of both the input data and the estimation error. The SSLMS algorithm belongs to the family of the LMS algorithm. The only difference in the filter weights update equations of these two algorithms is the application of the signum function on both the input data and the estimation error of the SSLMS algorithm.

The filter weights update equation of the SSLMS algorithm for real- and complex-valued data are given by (6.2) and (6.3), respectively [40], [41]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \, \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} \mathrm{sign}[e_i], \tag{6.2}$$

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \, \mathrm{csgn}[\mathbf{u}_i]^* \mathrm{csgn}[e_i]. \tag{6.3}$$

6.1.3 The SLMF Algorithm

The filter weights update equation of the Sign-Error Least Mean Fourth (SLMF) algorithm reduces to that of the Sign-Error Least Mean Square (SLMS) algorithm for both cases of real- and complex-valued data [46], [49].

Similarly, the filter weights update equations of the Sign-Sign Least Mean Fourth (SSLMF), Normalized Sign-Error Least Mean Fourth (NSLMF), and Normalized Sign-Sign Least Mean Fourth (NSSLMF) algorithms reduces to those of the Sign-Sign Least Mean Square (SSLMS), Normalized Sign-Error Least Mean Square (NSLMS), and Normalized Sign-Sign Least Mean Square (NSSLMS) algorithms, respectively, for both cases of real- and complex-valued data [46], [49].

6.1.4 The NSRLMS Algorithm

The Normalized Sign Regressor Least Mean Square (NSRLMS) algorithm is based on the clipping of the input data. The NSRLMS algorithm belongs to the family of the Normalized Least Mean Square (NLMS) algorithm [61]. The difference in the filter weights update equations of these two algorithms is the application of the signum function on the input data of the NSRLMS algorithm and the manner in which normalization has been applied.

The filter weights update equations of the NSRLMS algorithm for real- and complex-valued data are given by (6.4) and (6.5), respectively [42], [43]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2} \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i, \tag{6.4}$$

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2} \mathrm{csgn}[\mathbf{u}_i]^{*} e_i. \tag{6.5}$$

6.1.5 The NSLMS Algorithm

The Normalized Sign-Error Least Mean Square (NSLMS) algorithm is based on the clipping of the estimation error. The NSLMS algorithm belongs to the family of the NLMS algorithm [61]. The difference in the filter weights update equations of these two algorithms is the application of the signum function on the estimation error of the NSLMS algorithm and the manner in which normalization has been applied.

The filter weights update equations of the NSLMS algorithm for real- and complex-valued data are given by (6.6) and (6.7), respectively [44], [45]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2} \mathbf{u}_i^{\mathrm{T}} \mathrm{sign}[e_i], \tag{6.6}$$

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2} \mathbf{u}_i^{*} \mathrm{csgn}[e_i]. \tag{6.7}$$

In (6.1) to (6.7), $\mathbf{w}_i$ is the updated filter weight vector at iteration $i \geq 0$, μ is the step-size, ϵ is a small positive constant to avoid division by zero when the regressor is zero, $\mathbf{u}_i$ is the regressor vector, $e_i = d_i - y_i$ is the estimation error signal, d_i is the desired signal, y_i is the adaptive filter output, $\mathrm{sign}(.)$ denotes the sign of its argument, $\mathrm{csgn}(.)$ denotes the complex sign of its argument, the definitions of the signum function for real- and complex-valued data are given by (1.1) and (1.2), respectively, $||\mathbf{u}_i||_{\mathrm{H}}^2 = \mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, and $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined by:

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i1}|}, \frac{1}{|\mathbf{u}_{i2}|}, \dots, \frac{1}{|\mathbf{u}_{iM}|}\right\}, \tag{6.8}$$

where M is the filter length and $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$.

6.2 Background

6.2.1 The SRLMS Algorithm

The SRLMS algorithm is also frequently referred to as simply the Sign Regressor Algorithm (SRA) in the open literature. Some of the studies, which investigated the performance evaluation of the SRLMS algorithm are as follows. The stability of the SRLMS algorithm depends heavily on the characteristics of the input data [14]. It is shown in [14] that for some inputs, the LMS algorithm is stable while the SRLMS algorithm is unstable. In [15], the SRLMS algorithm with correlated Gaussian data was studied in the presence of both stationary and nonstationary environments.

In [24], the transient performance degradation of the SRLMS algorithm was studied for correlated input data. It was concluded that the SRLMS and LMS algorithms have the strongest degradation when compared to the various other algorithms [24]. In [62], the SRLMS algorithm was employed for the digital predistortion model identification purpose. It was shown that the application of the SRLMS algorithm for digital predistortion model identification can achieve similar linearization and convergence performance with much lower computational complexity when compared to the conventional least-square-based algorithm [62].

6.2.2 The SSLMS Algorithm

The SSLMS algorithm is also frequently referred to as simply the Sign-Sign Algorithm (SSA) in the open literature. Some of the studies, which investigated the performance evaluation of the SSLMS algorithm are as follows. The SSLMS algorithm with and without leakage was investigated in [26]. The convergence analysis of the SSLMS algorithm was performed in [28], [29]. Moreover, a rigorous tracking analysis of the SSLMS algorithm when employed in the identification of a time-varying plant with a white Gaussian input was performed in [31].

6.2.3 The SLMF Algorithm

The variants of the SLMF algorithm such as the NSLMF algorithm and the Block-Based Normalized Sign-Error Least Mean Fourth (BBNSLMF) algorithm were employed for removing multiple artifacts from the ElectroEncephaloGram (EEG) signal, namely power line noise, eye blink artifact, electromyogram, cardiac signal artifact, respiration artifact, and electrode motion artifact [55].

Similarly, the variants of the SSLMF algorithm such as the NSSLMF algorithm and the Block-Based Normalized Sign-Sign Least Mean Fourth (BBNSSLMF) algorithm were employed for removing the aforementioned multiple artifacts from the EEG signal [55].

It was concluded in [55] that the performance of the aforementioned algorithms, namely NSLMF, NSSLMF, BBNSLMF, BBNSSLMF, and various other algorithms analyzed in the paper is superior to the conventional Least Mean Fourth (LMF) algorithm. Hence these algorithms were found to be more suitable for remote health monitoring EEG system [55].

6.2.4 The NSRLMS Algorithm

The NSRLMS is also referred to as simply the Normalized Sign Regressor Algorithm (NSRA) in the open literature. Some of the studies, which investigated the performance evaluation of the NSRLMS algorithm are as follows. The NSRLMS algorithm was analyzed for both white Gaussian and colored Gaussian reference inputs in [63]. A fully analytical stochastic model for the NSRLMS algorithm for Gaussian inputs was presented in [64].

The NSRLMS algorithm was successfully employed for multiple artifacts reduction from the ElectroCardioGram (ECG) signal in [54], [65]. It was shown that the NSRLMS algorithm was the best performing algorithm for multiple artifacts reduction from the ECG signal among the six other algorithms studied in [54].

6.2.5 The NSLMS Algorithm

The NSLMS is also referred to as simply the Normalized Sign Algorithm (NSA) in the open literature. Some of the studies, which investigated the performance evaluation of the NSLMS algorithm are as follows. In [66], the convergence analysis of the NSLMS algorithm was performed and the algorithm was tested in an adaptive noise cancellation scenario. It was shown that the NSLMS algorithm performed better than the NLMS algorithm in cancelling the geomagnetic background noise in the desired signal of magnetic anomaly detection systems [66].

In another application, the NSLMS algorithm was successfully employed for multiple artifacts reduction from the ECG signal in [67]. It was shown that the NSLMS algorithm outperformed the traditional LMS algorithm in the cancellation of multiple artifacts from the ECG signal [67].

6.3 Contributions/Published Manuscripts

6.3.1 The SRLMS Algorithm

A published paper on the performance evaluation of the SRLMS [39] algorithm for complex-valued data is as follows:

[P10] **M. M. U. Faiz and A. Zerguine, "On the steady-state and tracking analysis of the complex SRLMS algorithm,"** ***in Proc. of the 22nd European Signal Processing Conf. (EUSIPCO 2014),*** **Lisbon, Portugal, pp. 751–754, Sep. 2014, E-ISBN: 978-0-9928-6261-9**

The SRLMS algorithm was analyzed and evaluated for the case of complex-valued data in [39]. The expression for the steady-state Mean Square Error (MSE) $\varphi = \mathrm{E}[|e_i|^2]$ of the complex SRLMS algorithm was derived and is given by (6.9) [39]:

$$\varphi = \frac{2\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2} - 2\mu \mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{6.9}$$

Also, the expression for the tracking MSE φ' of the complex SRLMS algorithm was derived and is given by (6.10) [39]:

$$\varphi' = \frac{4\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{2\sqrt{\pi\sigma_u^2} - 4\mu \mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{6.10}$$

In addition, the expression for the optimum step-size μ_{opt} of the complex SRLMS algorithm was also derived and is given by (6.11) [39]:

$$\mu_{\mathrm{opt}} = \frac{1}{2}\sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2}\left[\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2} + \frac{\sqrt{\pi\sigma_u^2}}{\mathrm{Tr}(\mathbf{R})}\right]} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2}. \tag{6.11}$$

In (6.9) to (6.11), $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ is the noise variance, $\sigma_u^2 = \mathrm{E}[|\mathbf{u}_i|^2]$ is the regressor variance, $\mathrm{Tr}(\mathbf{R})$ is the trace of the regressor covariance matrix $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^*\mathbf{u}_i]$, and $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^*]$ of the noise sequence $\mathbf{q}_i$. Moreover, it is shown that the simulation results are in a good match with the analytical results.

Finally, a comparison between the convergence performance of the complex SRLMS and complex LMS algorithms indicates slower convergence of the complex SRLMS algorithm for both white Gaussian and correlated Gaussian regressors in both Additive White Gaussian Noise (AWGN) and uniform noise environments with an SNR of 10 dB.

6.3.2 The SSLMS Algorithm

The two published papers on the performance evaluation of the SSLMS [40], [41] algorithm for real- and complex-valued data are as follows:

[P11] M. M. U. Faiz and A. Zerguine, "Steady-State and tracking analysis of the SSLMS algorithm," ***in Proc. of the 15th IEEE Int. Multi-Conf. on Systems, Signals & Devices (SSD 2018),*** **Hammamet, Tunisia, pp. 45–48, Mar. 2018, DOI: https://doi.org/10.1109/SSD.2018.8570395**

The SSLMS algorithm was analyzed and evaluated for the case of real-valued data in [40]. The expression for the steady-state MSE $\varphi = \mathrm{E}[e_i{}^2]$ of the SSLMS algorithm was derived and is given by (6.12) [40]:

$$\varphi = \frac{\mu \mathrm{Tr}(\mathbf{R})}{2\sigma_u}\left[\frac{\mu \mathrm{Tr}(\mathbf{R})}{4\sigma_u} + \sqrt{\frac{\mu^2[\mathrm{Tr}(\mathbf{R})]^2}{16\sigma_u^2} + \sigma_v^2}\right] + \sigma_v^2. \tag{6.12}$$

Also, the expression for the tracking MSE φ' of the SSLMS algorithm was derived and is given by (6.13) [40]:

$$\varphi' = \frac{\gamma}{4}\sqrt{\frac{\pi}{2}}\left[\frac{\gamma}{2}\sqrt{\frac{\pi}{2}} + \sqrt{\frac{\gamma^2\pi}{8} + 4\sigma_v^2}\right] + \sigma_v^2, \tag{6.13}$$

where $\gamma = \mu \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] + \mu^{-1}\mathrm{Tr}(\mathbf{Q})$. (6.14)

In addition, the expression for the optimum step-size μ_{opt} of the SSLMS algorithm was also derived and is given by (6.15) [40]:

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]}}. \tag{6.15}$$

In (6.12) to (6.15), $\sigma_v^2 = \mathrm{E}[v_i{}^2]$ is the noise variance, $\sigma_u^2 = \mathrm{E}[\mathbf{u}_i{}^2]$ is the regressor variance, $\mathrm{Tr}(\mathbf{R})$ is the trace of the regressor covariance matrix $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i]$, and $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$ of the noise sequence $\mathbf{q}_i$. Moreover, it is shown that the simulation results are in a good match with the analytical results.

Finally, a comparison between the convergence performance of the SSLMS and LMS algorithms indicates slower convergence of the SSLMS algorithm in a uniform noise environment with an SNR of 10 dB.

[P12] M. M. U. Faiz and A. Zerguine, "Analysis of the SSLMS algorithm for complex-valued data," *in Proc. of the 16th IEEE Int. Multi-Conf. on Systems, Signals & Devices (SSD 2019),* Istanbul, Turkey, pp. 262–265, Mar. 2019, DOI: https://doi.org/10.1109/SSD.2019.8893215

The SSLMS algorithm was analyzed and evaluated for the case of complex-valued data in [41]. The expression for the steady-state MSE $\varphi = \mathrm{E}[|e_i|^2]$ of the complex SSLMS algorithm was derived and is given by (6.16) [41]:

$$\varphi = \frac{2\mu \mathrm{Tr}(\mathbf{R})}{\sigma_u^2}\left[\mu \mathrm{Tr}(\mathbf{R}) + \sqrt{\mu^2[\mathrm{Tr}(\mathbf{R})]^2 + \sigma_u^2\sigma_v^2}\right] + \sigma_v^2. \tag{6.16}$$

Also, the expression for the tracking MSE φ' of the complex SSLMS algorithm was derived and is given by (6.17) [41]:

$$\varphi' = \frac{\gamma\sqrt{\pi}}{32}\left[\gamma\sqrt{\pi} + \sqrt{\gamma^2\pi + 64\sigma_v^2}\right] + \sigma_v^2, \qquad (6.17)$$

where $\gamma = 2\mu \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] + \mu^{-1}\mathrm{Tr}(\mathbf{Q})$. (6.18)

In addition, the expression for the optimum step-size μ_{opt} of the complex SSLMS algorithm was also derived and is given by (6.19) [41]:

$$\mu_{\text{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{2\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]}}. \qquad (6.19)$$

In (6.16) to (6.19), $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ is the noise variance, $\sigma_u^2 = \mathrm{E}[|\mathbf{u}_i|^2]$ is the regressor variance, $\mathrm{Tr}(\mathbf{R})$ is the trace of the regressor covariance matrix $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^*\mathbf{u}_i]$, and $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^*]$ of the noise sequence $\mathbf{q}_i$. Moreover, it is shown that the simulation results are in a good match with the analytical results.

Finally, a comparison between the convergence performance of the complex SSLMS and complex LMS algorithms indicates slower convergence of the complex SSLMS algorithm for both white Gaussian and correlated Gaussian regressors in both AWGN and uniform noise environments with an SNR of 10 dB.

6.3.3 The SLMF Algorithm

The two published papers on the performance evaluation of the SLMF [46], [49] algorithm and its variants for real- and complex-valued data are as follows:

[P13] M. M. U. Faiz and A. Zerguine, "Insights into the convergence and steady-state behaviors of the SLMF and its variants," ***in Proc. of the 12th IEEE Int. Multi-Conf. on Systems, Signals & Devices (SSD 2015),*** **Mahdia, Tunisia, pp. 1–4, Mar. 2015, DOI: https://doi.org/10.1109/SSD.2015.7348094**

In [46], it was shown that the filter weights update equations of the SLMF, SSLMF, NSLMF, and NSSLMF algorithms reduces to those of the SLMS, SSLMS, NSLMS, and NSSLMS algorithms, respectively, for both cases of real- and complex-valued data [46].

Moreover, it was also shown through rigorous simulations that the convergence and MSE performance of the SLMF, SSLMF, NSLMF, and NSSLMF algorithms are exactly the same as those of the SLMS, SSLMS, NSLMS, and NSSLMS algorithms, respectively, for both cases of real- and complex-valued data [46].

Finally, the author was the recipient of the Best Paper Award for the contribution in [46] (see Appendix C).

[P14] M. M. U. Faiz, "Comments on "Efficient signal conditioning techniques for brain activity in remote health monitoring network"," *IEEE Sensors Jour.,* vol. 15, no. 9, pp. 5349–5350, Sep. 2015, DOI: https://doi.org/10.1109/JSEN.2015.2431260

In [49], it was shown that the filter weights update equations of the SLMF and SSLMF algorithms for real-valued data are exactly identical to those of the SLMS and SSLMS algorithms, respectively.

Similarly, it can be shown that the filter weights update equations of all variants of the SLMF and SSLMF algorithms are exactly identical to those of the respective variants of the SLMS and SSLMS algorithms for both cases of real- and complex-valued data. For example, the filter weights update equations of the BBNSLMF and BBNSSLMF algorithms are exactly identical to those of the BBNSLMS and BBNSSLMS algorithms, respectively.

Therefore, it was concluded in [49] that the adaptive noise cancelers implemented using the NSLMF, BBNSLMF, NSSLMF, and BBNSSLMF algorithms in [55] for removing multiple artifacts from the EEG signal would deliver exactly the same performance compared to the adaptive noise cancelers implemented using the NSLMS, BBNSLMS, NSSLMS, and BBNSSLMS algorithms, respectively, provided the parameter settings and EEG data used are same.

6.3.4 The NSRLMS Algorithm

The two published papers on the performance evaluation of the NSRLMS [42], [43] algorithm for real- and complex-valued data are as follows:

[P15] M. M. U. Faiz and A. Zerguine, "The ε-Normalized Sign Regressor Least Mean Square (NSRLMS) adaptive algorithm," *in Proc. of the 2nd IEEE Int. Conf. on Signal and Image Processing Applications (ICSIPA 2011),* Kuala Lumpur, Malaysia, pp. 556–558, Nov. 2011, DOI: https://doi.org/10.1109/ICSIPA.2011.6144114

The NSRLMS algorithm was analyzed and evaluated for the case of complex-valued data in [42]. The expression for the steady-state MSE $\varphi = \mathrm{E}[|e_i|^2]$ of the complex NSRLMS algorithm was derived and is given by (6.20) [42]:

$$\varphi = \frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{(2-\mu)\sqrt{\pi\sigma_u^2}}\mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] + \sigma_v^2. \tag{6.20}$$

Also, the expression for the tracking MSE φ' of the complex NSRLMS algorithm was derived and is given by (6.21) [42]:

$$\varphi' = \frac{4\mathrm{Tr}(\mathbf{R})}{(2-\mu)\sqrt{\pi\sigma_u^2}}\left[\mu\sigma_v^2\mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\right] + \sigma_v^2. \tag{6.21}$$

In (6.20) and (6.21), $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ is the noise variance, $\sigma_u^2 = \mathrm{E}[|\mathbf{u}_i|^2]$ is the regressor variance, $\mathrm{Tr}(\mathbf{R})$ is the trace of the regressor covariance matrix $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^*\mathbf{u}_i]$, and $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^*]$ of the noise sequence $\mathbf{q}_i$. Moreover, it is shown that the simulation results are in a good match with the analytical results.

[P16] M. M. U. Faiz and A. Zerguine, "A note on NSRLMS, NSRLMF, and NSRLMMN adaptive algorithms," *in Proc. of the 15th IEEE Int. Multi-Conf. on Systems, Signals & Devices (SSD 2018),* Hammamet, Tunisia, pp. 40–44, Mar. 2018, DOI: https://doi.org/10.1109/SSD.2018.8570653

The NSRLMS algorithm was analyzed and evaluated for the case of real-valued data in [43] in order to compare the NSRLMS, NSRLMF, and NSRLMMN algorithms. The expression for the steady-state MSE $\varphi = \mathrm{E}[{e_i}^2]$ of the NSRLMS algorithm was derived and is given by (6.22) [43]:

$$\varphi = \frac{\mu\sigma_v^2}{2-\mu} + \sigma_v^2. \tag{6.22}$$

Also, the expression for the tracking MSE φ' of the NSRLMS algorithm was derived and is given by (6.23) [43]:

$$\varphi' = \frac{\mu\phi_1\sigma_v^2 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\phi_2 - \mu\phi_1} + \sigma_v^2, \tag{6.23}$$

$$\text{where } \phi_1 = \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{\left(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2\right)^2}\right], \tag{6.24}$$

$$\phi_2 = \mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{6.25}$$

In addition, the expression for the optimum step-size μ_{opt} of the NSRLMS algorithm was also derived and is given by (6.26) [43]:

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2\phi_1}\left[1 + \frac{\phi_1\mathrm{Tr}(\mathbf{Q})}{4\sigma_v^2\phi_2^2}\right]} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2\phi_2}. \tag{6.26}$$

In (6.22) to (6.26), $\sigma_v^2 = \mathrm{E}[{v_i}^2]$ is the noise variance and $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$ of the noise sequence $\mathbf{q}_i$. Moreover, it is shown that the simulation results are in a good match with the analytical results.

Finally, a comparison between the convergence performance of the NSRLMS, NSRLMF, and NSRLMMN algorithms indicates slower convergence of the NSRLMS algorithm in an AWGN environment with an SNR of 10 dB as expected.

6.3.5 The NSLMS Algorithm

The two published papers on the performance evaluation of the NSLMS [44], [45] algorithm for real- and complex-valued data are as follows:

[P17] M. M. U. Faiz and A. Zerguine, "A steady-state analysis of the ε-Normalized Sign-Error Least Mean Square (NSLMS) adaptive algorithm," ***in the Conf. Record of the 45th Asilomar Conf. on Signals, Systems, and Computers (Asilomar 2011),*** **Pacific Grove, CA, USA, pp. 538–541, Nov. 2011, DOI: https://doi.org/10.1109/ACSSC.2011.6190059**

The steady-state behavior of the NSLMS algorithm was analyzed and evaluated for both cases of real- and complex-valued data in [44]. The expression for the steady-state MSE $\varphi = \mathrm{E}[{e_i}^2]$ of the NSLMS algorithm was derived for real-valued data and is given by (6.27) [44]:

$$\varphi = \frac{\mu}{4}\sqrt{\frac{\pi}{2}}\left[\frac{\mu}{2}\sqrt{\frac{\pi}{2}} + \sqrt{\frac{\mu^2\pi}{8} + 4\sigma_v^2}\right] + \sigma_v^2, \tag{6.27}$$

where $\sigma_v^2 = \mathrm{E}[{v_i}^2]$ is the noise variance. In addition, the expression for the steady-state MSE $\varphi = \mathrm{E}[|e_i|^2]$ of the NSLMS algorithm was also derived for complex-valued data and is given by (6.28) [44]:

$$\varphi = \frac{\mu\sqrt{\pi}}{4}\left[\frac{\mu\sqrt{\pi}}{2} + \sqrt{\frac{\mu^2\pi}{4} + 4\sigma_v^2}\right] + \sigma_v^2, \tag{6.28}$$

where $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ is the noise variance. It is interesting to note that the expressions for the steady-state MSE of the NSLMS algorithm for real- and complex-valued data given in (6.27) and (6.28), respectively, are identical except for a scaling factor. Also, the steady-state MSE of the NSLMS algorithm for real- and complex-valued data is found to be independent of the regression data statistics. Moreover, it is shown that the simulation results are in a good match with the analytical results.

[P18] M. M. U. Faiz, A. Zerguine, S. M. Asad, and K. Mahmood, "Tracking MSE performance analysis of the ε-NSLMS algorithm," ***in Proc. of the 2nd IEEE Int. Conf. on Communications, Signal Processing and their Applications (ICCSPA 2015),*** **Sharjah, UAE, pp. 1–4, Feb. 2015, DOI: https://doi.org/10.1109/ICCSPA.2015.7081323**

The tracking behavior of the NSLMS algorithm was analyzed and evaluated for both cases of real- and complex-valued data in [45]. The expressions for the tracking MSE $\varphi' = \mathrm{E}[{e_i}^2]$ and optimum step-size μ_{opt} of the NSLMS algorithm were derived for real-valued data and are given by (6.29) and (6.30), respectively [45]:

$$\varphi' = \frac{\gamma\sqrt{\pi}}{4\phi_2^2}\left[\frac{\gamma\sqrt{\pi}}{4} + \sqrt{\frac{\gamma^2\pi}{16} + 2\sigma_v^2\phi_2^2}\right] + \sigma_v^2, \tag{6.29}$$

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\phi_1}}, \tag{6.30}$$

where $\gamma = \mu\phi_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})$, (6.31)

$$\phi_1 = \mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\epsilon+||\mathbf{u}_i||^2)^2}\right], \tag{6.32}$$

$$\phi_2 = \mathrm{E}\left[\frac{1}{\epsilon+||\mathbf{u}_i||^2}\right]. \tag{6.33}$$

In (6.29) to (6.33), $\sigma_v^2 = \mathrm{E}[{v_i}^2]$ is the noise variance and $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$ of the noise sequence $\mathbf{q}_i$.

Also, the expressions for the tracking MSE $\varphi' = \mathrm{E}[|e_i|^2]$ and optimum step-size of the NSLMS algorithm were derived for complex-valued data and are given by (6.34) and (6.35), respectively [45]:

$$\varphi' = \frac{\gamma\sqrt{\pi}}{8\phi_2^2}\left[\frac{\gamma\sqrt{\pi}}{4} + \sqrt{\frac{\gamma^2\pi}{16} + 4\sigma_v^2\phi_2^2}\right] + \sigma_v^2, \tag{6.34}$$

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{2\phi_1}}, \tag{6.35}$$

where $\gamma = 2\mu\phi_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})$. (6.36)

In (6.34) to (6.36), $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ is the noise variance and $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^*]$ of the noise sequence $\mathbf{q}_i$.

In addition, the expressions for the tracking MSE and optimum step-size of the NSLMS algorithm for both cases of real- and complex-valued data were also generalized in [45] as they are identical except for a scaling factor and are given by (6.37) and (6.38), respectively [45]:

$$\varphi' = \frac{\gamma\sqrt{\pi}}{4\alpha\phi_2^2}\left[\frac{\gamma\sqrt{\pi}}{4} + \sqrt{\frac{\gamma^2\pi}{16} + 2\alpha\sigma_v^2\phi_2^2}\right] + \sigma_v^2, \tag{6.37}$$

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\alpha\phi_1}}, \tag{6.38}$$

where $\gamma = \alpha\mu\phi_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})$, (6.39)

wherein the scaling factor α takes the value 1 and 2 for real- and complex-valued data cases, respectively. Moreover, it is shown that the simulation results are in a good match with the analytical results.

Finally, a comparison between the convergence performance of the NSLMS and NLMS algorithms indicates slower convergence of the NSLMS algorithm for both white Gaussian and correlated Gaussian regressors in an AWGN environment with an SNR of 10 dB. It is also observed that the convergence performance of the NSLMS algorithm gets more inferior compared to the NLMS algorithm for complex-valued data than real-valued data.

[P10]

ON THE STEADY-STATE AND TRACKING ANALYSIS OF THE COMPLEX SRLMS ALGORITHM

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ABSTRACT

In this paper, the steady-state and tracking behavior of the complex signed regressor least mean square (SRLMS) algorithm are analyzed in stationary and nonstationary environments, respectively. Here, the SRLMS algorithm is analyzed in the presence of complex-valued white and correlated Gaussian input data. Moreover, a comparison between the convergence performance of the complex SRLMS algorithm and the complex least mean square (LMS) algorithm is also presented. Finally, simulation results are presented to support our analytical findings.

Index Terms— LMS, SRLMS, Steady-state, Tracking.

1. INTRODUCTION

Computational complexity reduction of the adaptive noise cancelation system, particularly, in applications such as wireless biotelemetry system is very important [1]. The signed regressor least mean square (SRLMS) algorithm is known to have a reduced computational complexity compared to that of the traditional least mean square (LMS) algorithm [2]. Therefore, adaptive filters equipped with the signed versions of the LMS algorithm (such as the SRLMS algorithm) are extensively used for the processing and analysis of electrocardiogram (ECG) signals [1].

The SRLMS algorithm is obtained from the conventional LMS algorithm by replacing the regressor vector by its sign. The SRLMS algorithm is also referred to as simply the signed regressor algorithm (SRA) [2]–[3]. Theoretical studies of the SRLMS algorithm can be found in [2]–[5]. To the best of the authors knowledge, the steady-state and tracking analysis of the SRLMS algorithm for the case of complex-valued data are not available in the literature of adaptive filtering. Therefore, this work reports the findings of the steady-state and tracking analysis of the SRLMS algorithm for the case of complex-valued data.

The organization of the paper is as follows. The complex SRLMS algorithm is described briefly in Section 2. The steady-state and tracking analysis of the complex SRLMS algorithm are derived in Sections 3 and 4, respectively. Finally, simulation results and some concluding remarks are presented in Sections 5 and 6, respectively.

2. THE COMPLEX SRLMS ALGORITHM

The weight update recursion for the complex SRLMS algorithm is governed by

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \operatorname{csgn}[\mathbf{u}_i]^* e_i, \quad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ is the updated weight vector, μ is the step-size, $\mathbf{u}_i$ is the regressor vector, and e_i denotes the estimation error given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}, \tag{2}$$

where d_i is the desired value.

3. STEADY-STATE ANALYSIS

We shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the stationary data model [6]:

A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that

$$d_i = \mathbf{u}_i \mathbf{w}^o + v_i. \tag{3}$$

A.2 The additive noise sequence v_i is independent and identically distributed (i.i.d.) circular with variance $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ and is independent of $\mathbf{u}_j$ for all i, j.

A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.

A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}\left[\mathbf{u}_i^* \mathbf{u}_i\right] > \mathbf{0}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [6]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right] = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]\right], \quad \text{as} \quad i \to \infty, \tag{4}$$

where

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \mathrm{E}[\mathrm{Re}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i] \mathbf{u}_i^*]], \tag{5}$$

$$e_i = e_{a_i} + v_i, \tag{6}$$

with $\mathrm{H}[\mathbf{u}_i]$ denoting some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$, $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error, $g[e_i]$ denotes some function of e_i and for the complex SRLMS algorithm $g[e_i] = e_i$. Then, by using the fact that e_{a_i} is independent of v_i, we reach at the following expression for the term $\mathrm{E}\left[e_{a_i}^* g[e_i]\right]$:

$$\mathrm{E}\left[e_{a_i}^* g[e_i]\right] = \mathrm{E}[|e_{a_i}|^2]. \tag{7}$$

To evaluate the term $\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |g[e_i]|^2\right]$, we start by noting that

$$|g[e_i]|^2 = |e_{a_i}|^2 + |v_i|^2 + e_{a_i}^* v_i + e_{a_i} v_i^*. \tag{8}$$

Now, if we multiply (8) by $||\mathbf{u}_i||_{\mathrm{H}}^2$ from the left, then take the expected value of the resulting equation and use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, we obtain

$$\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |g[e_i]|^2\right] = \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2 |e_{a_i}|^2] + \sigma_v^2 \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]. \tag{9}$$

In [7], we have shown that

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \frac{4\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}}. \tag{10}$$

Substituting (10) into (9) we get

$$\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |g[e_i]|^2\right] = \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2 |e_{a_i}|^2] + \frac{4\sigma_v^2\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}}. \tag{11}$$

Substituting (7) and (11) into (4) we get

$$\mu\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2 |e_{a_i}|^2] + \frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}} = 2\mathrm{E}[|e_{a_i}|^2]. \tag{12}$$

In order to simplify (12), we use the separation principle, namely, that at steady-state, $||\mathbf{u}_i||_{\mathrm{H}}^2$ is independent of $e_{a_i}^2$. Therefore, we obtain

$$\frac{4\mu\mathrm{Tr}(\mathbf{R})\mathrm{E}[|e_{a_i}|^2]}{\sqrt{\pi\sigma_u^2}} + \frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}} = 2\mathrm{E}[|e_{a_i}|^2]. \tag{13}$$

This leads to the expression for the steady-state excess-mean-square error (EMSE), $\zeta = \mathrm{E}[|e_{a_i}|^2]$, of the complex SRLMS algorithm which is given by

$$\zeta = \frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{2\sqrt{\pi\sigma_u^2} - 4\mu\mathrm{Tr}(\mathbf{R})}. \tag{14}$$

Ultimately, the steady-state mean-square error (MSE), $\varphi = \mathrm{E}\left[|e_i|^2\right]$, of the complex SRLMS algorithm is given by

$$\varphi = \frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{2\sqrt{\pi\sigma_u^2} - 4\mu\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{15}$$

4. TRACKING ANALYSIS

Here, let us assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the nonstationary data model [6]:

A.5 There exists an optimal weight vector $\mathbf{w}_i^o$ such that

$$d_i = \mathbf{u}_i \mathbf{w}_i^o + v_i. \tag{16}$$

A.6 The weight vector varies according to the random-walk model

$$\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i, \tag{17}$$

where the Gaussian noise sequence $\mathbf{q}_i$ is i.i.d. with variance σ_q^2 and covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.

A.7 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

In this case, the following variance relation holds [6]:

$$\mu\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |g[e_i]|^2\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^* g[e_i]\right]\right], \quad \text{as } i \to \infty. \tag{18}$$

Therefore, by substituting (7) and (11) into (18), the tracking EMSE of the complex SRLMS algorithm can be shown to be

$$\zeta = \frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R}) + \sqrt{\pi\sigma_u^2}\,\mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\sqrt{\pi\sigma_u^2} - 4\mu\mathrm{Tr}(\mathbf{R})}. \tag{19}$$

The optimum value of the step-size of the complex SRLMS algorithm can be obtained by minimizing (19) with respect to μ and is given by

$$\mu_{\mathrm{opt}} = \frac{1}{2}\sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2}\left[\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2} + \frac{\sqrt{\pi\sigma_u^2}}{\mathrm{Tr}(\mathbf{R})}\right]} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2}. \tag{20}$$

Ultimately, the corresponding minimum value of the tracking MSE of the complex SRLMS algorithm is given by

$$\varphi_{\min} = \frac{4\mu_{\mathrm{opt}}\sigma_v^2\mathrm{Tr}(\mathbf{R}) + \sqrt{\pi\sigma_u^2}\,\mu_{\mathrm{opt}}^{-1}\mathrm{Tr}(\mathbf{Q})}{2\sqrt{\pi\sigma_u^2} - 4\mu_{\mathrm{opt}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{21}$$

Finally, Table 1 and Table 2 report the expressions for the steady-state EMSE and the tracking EMSE of the complex SRLMS and LMS algorithms, respectively.

Table 1. Performance comparison of the steady-state EMSE for the LMS and the complex SRLMS algorithms.

Algorithm	**Steady-state EMSE**
LMS	$\frac{\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{2-\mu\mathrm{Tr}(\mathbf{R})}$
Complex SRLMS	$\frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{2\sqrt{\pi\sigma_u^2}-4\mu\mathrm{Tr}(\mathbf{R})}$

5. SIMULATION RESULTS

Several simulations are carried out in order to corroborate our theoretical findings. In all the simulations, the filter length is fixed at $M = 5$. In Fig. 2 and Fig. 4, the correlated data is obtained by passing a unit-variance i.i.d. Gaussian data through a first-order auto-regressive model with transfer function $\frac{\sqrt{1-a^2}}{(1-az^{-1})}$ and $a = 0.8$. In Figures 3–4, we have chosen $\sigma_q^2 = 10^{-8}$. Additive white Gaussian noise (AWGN)

Table 2. Performance comparison of the tracking EMSE for the LMS and the complex SRLMS algorithms.

Algorithm	**Tracking EMSE**
LMS	$\frac{\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})+\mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2-\mu\mathrm{Tr}(\mathbf{R})}$
Complex SRLMS	$\frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})+\sqrt{\pi\sigma_u^2}\,\mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\sqrt{\pi\sigma_u^2}-4\mu\mathrm{Tr}(\mathbf{R})}$

environment is considered in Figures 1–6, while in Figures 7–8 uniform noise environment is considered. The signal-to-noise ratio (SNR) is fixed at 30 dB in Figures 1–4 and 10 dB in Figures 5–8.

First, the steady-state MSE of the complex SRLMS algorithm using white and correlated Gaussian regressors is shown in Figures 1–2, respectively. As can be seen from these figures, the simulation results are in a very good match with the theoretical result in equation (15) for values of μ ranging from 0.0001 to 0.01.

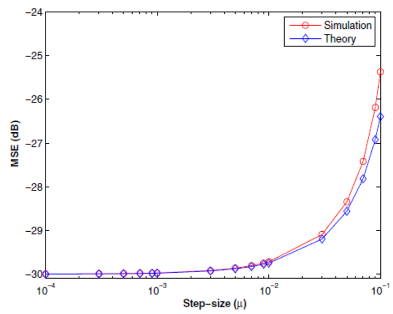

Fig. 1. Theoretical and simulated steady-state MSE of the SRLMS algorithm using white Gaussian regressors.

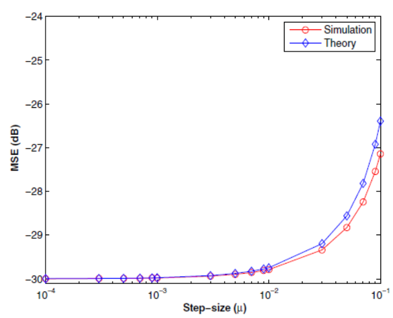

Fig. 2. Theoretical and simulated steady-state MSE of the SRLMS algorithm using correlated Gaussian regressors.

Second, Figures 3–4 demonstrate the tracking performance of the complex SRLMS algorithm using white and correlated Gaussian regressors, respectively. A zoom into the region around $\mu = 0.002$ shows that the tracking MSE possesses a minimum value of -29.828394 in Fig. 3 and -29.871226 in Fig. 4 at $\mu = 0.003$, which are in very good agreement with the corresponding theoretical values of $\varphi_{\min} = -29.896845$ and $\mu_{\mathrm{opt}} = 0.00208$ obtained from expressions (21) and (20), respectively.

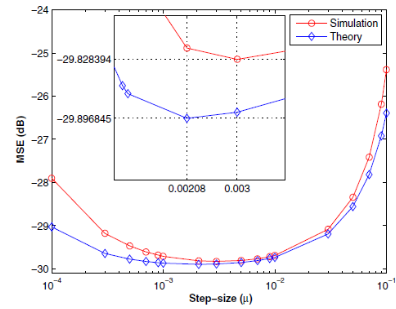

Fig. 3. Theoretical and simulated tracking MSE of the SRLMS algorithm using white Gaussian regressors.

Finally, the convergence behavior of the complex SRLMS algorithm is compared to that of the complex LMS algorithm in an unknown system identification setup. Figures 5 and 7 show the convergence performance of both the algorithms using white Gaussian regressors, while Fig. 6 and Fig. 8 show the convergence comparison using correlated Gaussian regressors. As observed from these figures, the complex LMS algorithm results in superior performance over the complex SRLMS algorithm for the same misadjustment.

6. CONCLUSIONS

In this work, analytical expressions are derived for the steady-state MSE, optimal step-size, and the corresponding optimal tracking MSE of the SRLMS algorithm for complex-valued data case. We observed that the theoretical values of the op-

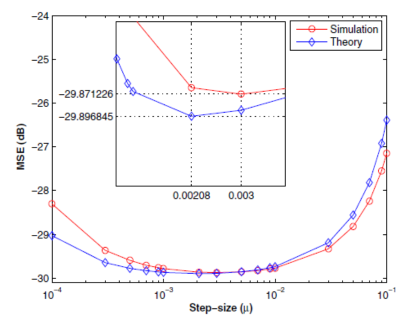

Fig. 4. Theoretical and simulated tracking MSE of the SRLMS algorithm using correlated Gaussian regressors.

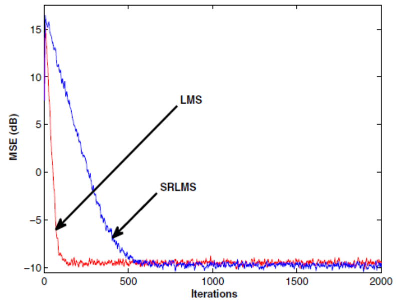

Fig. 5. Convergence comparison of the LMS and the SRLMS algorithms using white Gaussian regressors in an AWGN environment.

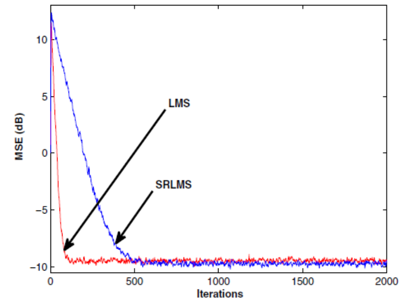

Fig. 7. Convergence comparison of the LMS and the SRLMS algorithms using white Gaussian regressors in a uniform noise environment.

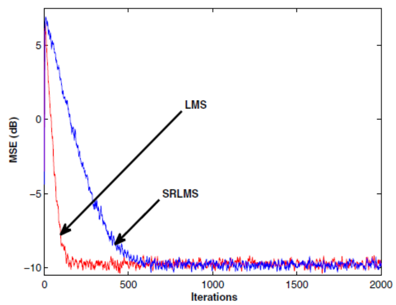

Fig. 6. Convergence comparison of the LMS and the SRLMS algorithms using correlated Gaussian regressors in an AWGN environment.

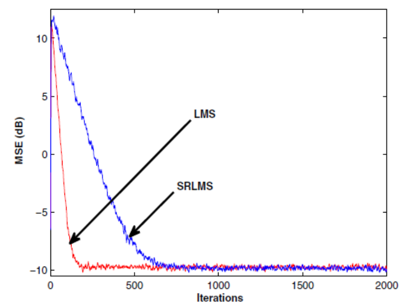

Fig. 8. Convergence comparison of the LMS and the SRLMS algorithms using correlated Gaussian regressors in a uniform noise environment.

timal step-size and the resulting minimum MSE of the complex SRLMS algorithm are similar for white and correlated Gaussian data. Furthermore, we also observed that the theoretical and simulation values of the optimal MSE of the complex SRLMS algorithm are in much closer agreement for correlated Gaussian data than white Gaussian data. Finally, as expected, the complex SRLMS algorithm has been shown to exhibit slower convergence rate than the complex LMS algorithm for the same misadjustment.

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Steady-State and Tracking Analysis of the SSLMS Algorithm

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***Abstract*—This paper presents expressions for the steady-state mean-square error (MSE), the optimum step-size, and the corresponding minimum value of the tracking MSE of the sign-sign least mean square (SSLMS) algorithm for the case of real-valued data. Then, simulation results are presented to support our analytical findings and found to corroborate them very well. Also, the performance of the SSLMS algorithm is compared to that of the LMS algorithm in the case where the noise statistics are uniformly distributed.**

***Index Terms*—SSA, SSLMS, steady-state, tracking.**

I. Introduction

The sign-sign least mean square (SSLMS) algorithm or simply the sign-sign algorithm (SSA) is based on clipping of both the input data and the estimation error. The SSLMS algorithm has been widely studied over the past years [1]– [9].

In [10], a unified approach based on energy conservation relation [11] was used to study the steady-state and tracking analysis of a number of adaptive filters. To the best of the authors' knowledge, the steady-state and tracking analysis of the SSLMS algorithm based on this unified approach are not available in the literature of adaptive filtering. Therefore, this work reports the findings of the steady-state and tracking analysis of the SSLMS algorithm based on this unified approach.

The rest of the paper is organized as follows. The SSLMS algorithm is described in Section 2. The steady-state and tracking analysis of the SSLMS algorithm is carried out in Sections 3 and 4, respectively. The simulation results are discussed in Section 5. Finally, some concluding remarks are reported in Section 6.

II. The SSLMS Algorithm

The SSLMS algorithm is based on clipping of both the regressor vector $\mathbf{u}_i$ with variance $\sigma_u^2 = \mathrm{E}[\mathbf{u}_i^2]$ and the estimation error e_i. The SSLMS algorithm updates its weight vector $\mathbf{w}_i$ according to the following recursive rule [1]– [9]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \ \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} \mathrm{sign}[e_i], \qquad i \geq 0, \qquad (1)$$

where μ is the step-size, $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, and $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined for filter length M as

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{ \frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i_M}|} \right\}. \qquad (2)$$

III. Steady-State Analysis

To carry out the steady-state analysis of the SSLMS algorithm we shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the stationary data model [10]– [12]:

A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that $d_i = \mathbf{u}_i \mathbf{w}^o + v_i$.

A.2 The noise sequence v_i is independent and identically distributed (i.i.d.) with variance $\sigma_v^2 = \mathrm{E}[v_i^2]$ and is independent of $\mathbf{u}_j$ for all i, j.

A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.

A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_i] > \mathbf{0}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [10]– [11]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 \mathrm{g}^2[e_i] \right] = 2\mathrm{E}\left[e_{a_i} \mathrm{g}[e_i] \right], \ \text{as} \ i \to \infty, \quad (3)$$

where

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \mathrm{E}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i] \mathbf{u}_i^{\mathrm{T}}], \qquad (4)$$

$$e_i = e_{a_i} + v_i, \qquad (5)$$

with $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ denoting the a priori estimation error, $\mathrm{g}[e_i]$ denoting some function of e_i, and for the SSLMS algorithm $\mathrm{g}[e_i]$ can be written as

$$\mathrm{g}[e_i] = \mathrm{sign}[e_{a_i} + v_i]. \qquad (6)$$

Substituting (6) into (3) and by using the fact that $(\mathrm{sign}[x])^2 = 1$, we get

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2\right] = 2\mathrm{E}\left[e_{a_i}\mathrm{sign}[e_{a_i} + v_i]\right]. \quad (7)$$

From [12], we have

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}), \quad (8)$$

where $\mathrm{Tr}(\mathbf{R})$ is the trace of $\mathbf{R}$. From [13], we have

$$\mathrm{E}\left[e_{a_i}\mathrm{sign}[e_{a_i} + v_i]\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\mathrm{E}[e_{a_i}^2]}{\sqrt{\mathrm{E}[e_{a_i}^2] + \sigma_v^2}}\right]. \quad (9)$$

By substituting (8) and (9) into (7), the expression for the steady-state excess-mean-square error (EMSE), $\zeta = \mathrm{E}[e_{a_i}^2]$, of the SSLMS algorithm can be shown to be

$$\zeta = \frac{\mu \mathrm{Tr}(\mathbf{R})}{2\sigma_u}\left[\frac{\mu \mathrm{Tr}(\mathbf{R})}{4\sigma_u} + \sqrt{\frac{\mu^2[\mathrm{Tr}(\mathbf{R})]^2}{16\sigma_u^2} + \sigma_v^2}\right]. \quad (10)$$

Finally, the expression for the steady-state mean-square error (MSE), $\varphi = \mathrm{E}\left[e_i^2\right]$, of the SSLMS algorithm is given by

$$\varphi = \zeta + \sigma_v^2. \quad (11)$$

IV. Tracking Analysis

To carry out the tracking analysis of the SSLMS algorithm we shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the nonstationary data model [10], [11], [14]:

- A.5 There exists a vector $\mathbf{w}_i^o$ such that $d_i = \mathbf{u}_i\mathbf{w}_i^o + v_i$.
- A.6 The weight vector varies according to the random-walk model $\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i$, and the noise sequence $\mathbf{q}_i$ is i.i.d. with variance $\sigma_q^2 = \mathrm{E}[\mathbf{q}_i^2]$ and covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.
- A.7 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [10]– [11]:

$$\begin{aligned}&\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) \\ &\quad = 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \text{ as } i \to \infty. \quad (12)\end{aligned}$$

Substituting (6) into (12) and by using the fact that $(\mathrm{sign}[x])^2 = 1$, we get

$$\gamma = 2\mathrm{E}\left[e_{a_i}\mathrm{sign}[e_{a_i} + v_i]\right], \quad (13)$$

which can be set up as follows:

$$\gamma = \mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}). \quad (14)$$

By substituting (9) into (13) the expression for the tracking EMSE ζ' of the SSLMS algorithm can be shown to be

$$\zeta' = \frac{\gamma}{4}\sqrt{\frac{\pi}{2}}\left[\frac{\gamma}{2}\sqrt{\frac{\pi}{2}} + \sqrt{\frac{\gamma^2\pi}{8} + 4\sigma_v^2}\right]. \quad (15)$$

Consequently, the optimum step-size μ_{opt} of the SSLMS algorithm can be obtained by minimizing (14) with respect to μ and is given by

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2\right]}}. \quad (16)$$

Finally, the corresponding minimum value of the tracking MSE $\varphi'_{\min}$ of the SSLMS algorithm is given by

$$\varphi'_{\min} = \frac{\gamma_{\mathrm{opt}}}{4}\sqrt{\frac{\pi}{2}}\left[\frac{\gamma_{\mathrm{opt}}}{2}\sqrt{\frac{\pi}{2}} + \sqrt{\frac{\gamma_{\mathrm{opt}}^2\pi}{8} + 4\sigma_v^2}\right] + \sigma_v^2, \quad (17)$$

where

$$\gamma_{\mathrm{opt}} = \mu_{\mathrm{opt}}\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2\right] + \mu_{\mathrm{opt}}^{-1}\mathrm{Tr}(\mathbf{Q}). \quad (18)$$

V. Simulation Results

In this section, the simulation results are reported as follows: Uniform noise is considered in Figure 1 whereas additive white Gaussian noise (AWGN) is considered in Figures 2–5. M is fixed at 5 in Figure 1 and at 10 in Figures 2–5. The signal-to-noise ratio (SNR) is fixed at 10 dB in Figure 1, at 30 dB in Figures 2–3, and at 20 dB in Figures 4–5. Finally, σ_q^2 is fixed at 10^{-8} in Figures 4–5.

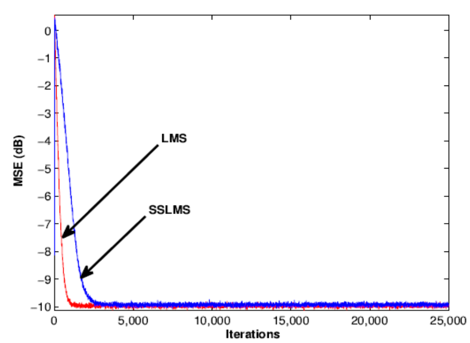

Fig. 1. Comparison of the convergence curves of the SSLMS and LMS algorithms.

As can be seen from Figure 1, the convergence performance of the SSLMS algorithm is found to be inferior when compared with to that of the LMS algorithm. The LMS algorithm converges faster than the SSLMS algorithm in this scenario. In Figures 2–3, the steady-state MSE of the SSLMS algorithm is plotted for varying step-sizes and different regressors. As can be seen from these figures, the simulation result and the theoretical result in (11) are in close agreement for correlated Gaussian regressors than white Gaussian regressors.

Finally, Figures 4–5 show the tracking performance of the SSLMS algorithm using white and correlated Gaussian regressors, respectively. A zoom into the region around $\mu = 0.0001$ in Figure 4 shows that the tracking MSE possesses a minimum value of -19.9314948 dB at $\mu = 0.0001$, which are in good agreement with the corresponding theoretical values of $\varphi'_{\text{min}} = -19.9513803$ dB at $\mu_{\text{opt}} = 0.0001119$ obtained from expressions (17) and (16), respectively. Similarly, a zoom into the region around $\mu = 0.0001$ in Figure 5 shows that the tracking MSE possesses a minimum value of -19.9484745 dB at $\mu = 0.0001$, which are in very good agreement with the corresponding theoretical values of $\varphi'_{\text{min}} = -19.9513803$ dB at $\mu_{\text{opt}} = 0.0001119$ obtained from expressions (17) and (16), respectively.

VI. Conclusions

In this work, expressions are derived for the steady-state MSE, the optimum step-size, and the corresponding minimum value of the tracking MSE of the SSLMS algorithm for the case of real-valued data. It is shown that the convergence performance of the SSLMS algorithm is inferior when compared with that of the LMS algorithm. Finally, we observe a close agreement between the simulation and analytical results for correlated Gaussian regressors than for white Gaussian regressors.

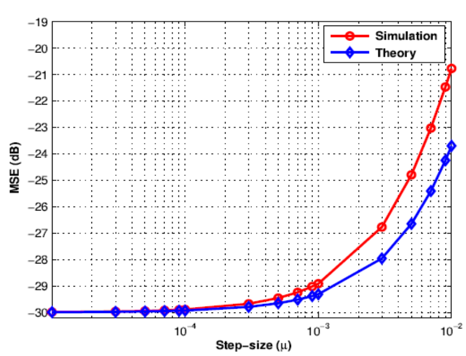

Fig. 2. Theoretical and simulated steady-state MSE of the SSLMS algorithm using white Gaussian regressors.

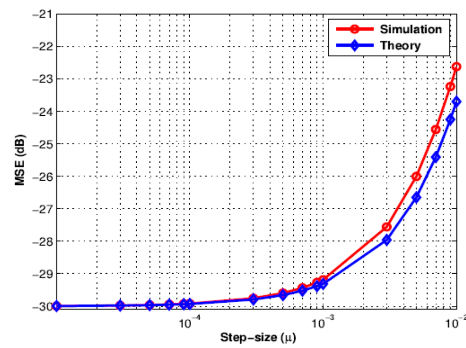

Fig. 3. Theoretical and simulated steady-state MSE of the SSLMS algorithm using correlated Gaussian regressors.

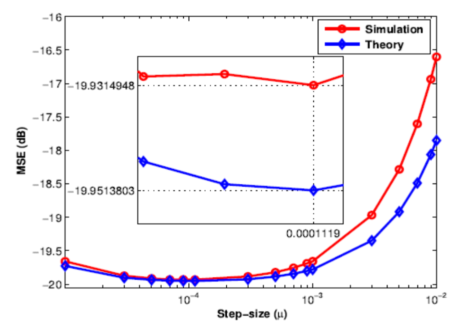

Fig. 4. Theoretical and simulated tracking MSE of the SSLMS algorithm using white Gaussian regressors.

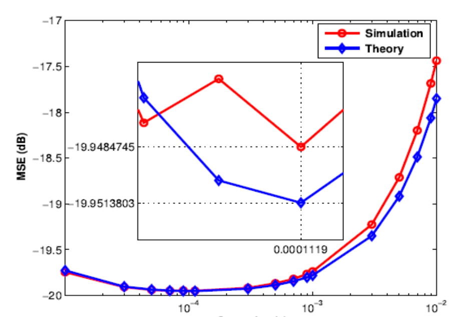

Fig. 5. Theoretical and simulated tracking MSE of the SSLMS algorithm using correlated Gaussian regressors.

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[P12]

Analysis of the SSLMS algorithm for complex-valued data

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***Abstract*—Terms for the steady-state mean-square error, the optimum step-size, and the corresponding minimum value of the tracking mean-square error of the sign–sign least mean square algorithm for complex-valued data are derived in the work. Moreover, convergence comparison of the least mean square (LMS) algorithm and the sign–sign LMS algorithm for complex-valued data is also presented. Finally, simulation results are presented to support our analytical findings.**

***Index Terms*—Complex-valued data, sign–sign LMS, Tracking, Steady-State.**

I. Introduction

The presented sign–sign least mean square (SSLMS) algorithm is known to have a reduced computational complexity compared to that of the conventional least mean square (LMS) algorithm [1]– [9]. Therefore, adaptive filters equipped with the signed versions of the LMS algorithm (such as the sign–sign LMS algorithm) are extensively used for the processing and analysis of electrocardiogram (ECG) signals [10].

The sign–sign LMS algorithm is based on clipping the regressor vector and the estimated error. The proposed sign–sign LMS algorithm carries the adaptation of its weight vector $\mathbf{w}_i$ according to the following recursive rule:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \ \mathrm{csgn}[\mathbf{u}_i]^* \mathrm{csgn}[e_i], \quad i \geq 0, \tag{1}$$

where μ is the step-size, $\mathbf{u}_i$ is the regressor vector with variance $\sigma_u^2 = \mathrm{E}[|\mathbf{u}_i|^2]$, and e_i is the estimation error given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}, \tag{2}$$

with d_i denoting the desired value. This work reports the findings of the steady-state and tracking analysis of the sign–sign LMS algorithm for complex-valued data based on energy conservation approach [11]– [12].

The rest of the paper is organized as follows: The steady-state analysis is carried out in Section II. In Section III, the tracking analysis of the proposed algorithm is studied, while the performance of the filter is tested in Section IV. Finally, Section V concludes the paper.

II. Steady-State Analysis

This section reports the steady-state analysis of of the sign–sign LMS algorithm. Hence, for this task it shall be assumed that the data $\{d_i, \mathbf{u}_i\}$ to satisfy the following assumptions of the stationary data model [11]– [12]:

A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that

$$d_i = \mathbf{u}_i \mathbf{w}^o + v_i. \tag{3}$$

A.2 The additive noise sequence v_i is independent and identically distributed (i.i.d.) circular with variance $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ and is independent of $\mathbf{u}_j$ for all i, j.

A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.

A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}\left[\mathbf{u}_i^* \mathbf{u}_i\right] > \mathbf{0}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [11]– [12]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}} |\mathrm{g}[e_i]|^2\right] = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]\right], \quad \text{as} \quad i \to \infty, \tag{4}$$

where

$$\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}] \quad = \quad \mathrm{E}[\mathrm{Re}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i] \mathbf{u}_i^*]], \tag{5}$$

and

$$e_i \quad = \quad e_{a_i} + v_i, \tag{6}$$

with $\mathrm{H}[\mathbf{u}_i]$ denoting some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$, the a priori estimation error is given by

$$e_{a_i} = \mathbf{u}_i (\mathbf{w}^o - \mathbf{w}_{i-1}) \tag{7}$$

$\mathrm{g}[e_i]$ denotes a function of e_i and for the SSLMS algorithm $\mathrm{g}[e_i]$ is defined as

$$\mathrm{g}[e_i] = \mathrm{csgn}[e_{a_i} + v_i]. \tag{8}$$

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Substituting (8) into (4), and using the fact that $|\mathrm{csgn}[x]|^2 = 2$ almost everywhere in the complex plane, we get

$$\mu \mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\right] = \mathrm{Re}\left[\mathrm{E}\left[e^*_{a_i}\mathrm{csgn}[e_{a_i} + v_i]\right]\right]. \quad (9)$$

In [13], we have shown that

$$\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}] = \frac{4\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma^2_u}}, \quad (10)$$

with the trace of $\mathbf{R}$ given by $\mathrm{Tr}(\mathbf{R})$. From [14], then we can write

$$\mathrm{Re}\left[\mathrm{E}\left[e^*_{a_i}\mathrm{csgn}[e_{a_i} + v_i]\right]\right] = \frac{2}{\sqrt{\pi}}\left(\frac{\mathrm{E}[|e_{a_i}|^2]}{\sqrt{\mathrm{E}[|e_{a_i}|^2] + \sigma^2_v}}\right).$$

The steady-state excess-mean-square error (EMSE) expression for the SSLMS algorithm, $\zeta = \mathrm{E}[|e_{a_i}|^2]$, is derived after substituting (10) and (11) into (9), and is given by

$$\zeta = \frac{2\mu\mathrm{Tr}(\mathbf{R})}{\sigma^2_u}\left[\mu\mathrm{Tr}(\mathbf{R}) + \sqrt{\mu^2[\mathrm{Tr}(\mathbf{R})]^2 + \sigma^2_u\sigma^2_v}\right]. \quad (11)$$

Ultimately, the steady-state mean-square error (MSE), $\varphi = \mathrm{E}\left[|e_i|^2\right]$, of the sign–sign LMS algorithm is given by

$$\varphi = \zeta + \sigma^2_v. \quad (12)$$

III. Tracking Analysis

To carry out the tracking analysis of the sign–sign LMS algorithm we shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the nonstationary data model [11]– [12]:

- A.5 There exists a vector $\mathbf{w}^o_i$ such that $d_i = \mathbf{u}_i\mathbf{w}^o_i + v_i$.
- A.6 The weight vector varies according to the random-walk model $\mathbf{w}^o_i = \mathbf{w}^o_{i-1} + \mathbf{q}_i$, where the Gaussian noise sequence $\mathbf{q}_i$ is independent and identically distributed with $\sigma^2_q = \mathrm{E}[|\mathbf{q}_i|^2]$ and covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}^*_i]$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.
- A.7 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}^o_{-1}\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [11]– [12] when $i \to \infty$:

$$\mu\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}|\mathrm{g}[e_i]|^2\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{Re}\left[\mathrm{E}\left[e^*_{a_i}\mathrm{g}[e_i]\right]\right]. \quad (13)$$

Substituting (8) into (13) and knowing that $|\mathrm{csgn}[x]|^2 = 2$, we obtain

$$\gamma = 2\mathrm{Re}\left[\mathrm{E}\left[e^*_{a_i}\mathrm{csgn}[e_{a_i} + v_i]\right]\right], \quad (14)$$

where

$$\gamma = 2\mu\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}). \quad (15)$$

By substituting (11) into (14) the tracking EMSE of the SSLMS adaptive algorithm results in

$$\zeta' = \frac{\gamma\sqrt{\pi}}{32}\left[\gamma\sqrt{\pi} + \sqrt{\gamma^2\pi + 64\sigma^2_v}\right]. \quad (16)$$

Consequently, the optimum step-size μ_{opt} of the sign–sign LMS algorithm can be obtained by minimizing (15) with respect to μ and is given by

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{2\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\right]}}. \quad (17)$$

Finally, the corresponding minimum value of the tracking MSE $\varphi'_{\min}$ of the sign–sign LMS algorithm is given by

$$\varphi'_{\min} = \frac{\gamma_{\mathrm{opt}}\sqrt{\pi}}{32}\left[\gamma_{\mathrm{opt}}\sqrt{\pi} + \sqrt{\gamma^2_{\mathrm{opt}}\pi + 64\sigma^2_v}\right] + \sigma^2_v, \quad (18)$$

where

$$\gamma_{\mathrm{opt}} = 2\mu_{\mathrm{opt}}\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\right] + \mu^{-1}_{\mathrm{opt}}\mathrm{Tr}(\mathbf{Q}). \quad (19)$$

Finally, Tables I and II report the expressions for the steady-state EMSE and tracking EMSE of the LMS and complex sign–sign LMS algorithms, respectively.

TABLE I
STEADY-STATE EMSE EXPRESSIONS COMPARISON OF THE LMS AND COMPLEX SIGN–SIGN LMS ALGORITHMS.

Algorithm	Steady-state EMSE Expression
LMS	$\frac{\mu\sigma^2_v\mathrm{Tr}(\mathbf{R})}{2-\mu\mathrm{Tr}(\mathbf{R})}$
Complex SSLMS	$\frac{2\mu^2(\mathrm{Tr}(\mathbf{R}))^2}{\sigma^2_u} + \frac{2\mu\mathrm{Tr}(\mathbf{R})}{\sigma^2_u}\sqrt{\mu^2[\mathrm{Tr}(\mathbf{R})]^2 + \sigma^2_u\sigma^2_v}$

TABLE II
TRACKING EMSE EXPRESSIONS COMPARISON OF THE LMS AND COMPLEX SIGN–SIGN LMS ALGORITHMS.

Algorithm	Tracking EMSE Expression
LMS	$\frac{\mu\sigma^2_v\mathrm{Tr}(\mathbf{R})+\mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2-\mu\mathrm{Tr}(\mathbf{R})}$
Complex sign–sign LMS	$\frac{\gamma\sqrt{\pi}}{32}\left[\gamma\sqrt{\pi} + \sqrt{\gamma^2\pi + 64\sigma^2_v}\right]$

IV. Simulation Results

In Figures 1–4, the filter length is fixed at $M = 10$ and signal-to-noise ratio (SNR) is fixed at 10 dB. In Figures 5–8, the filter length is fixed at $M = 5$ and SNR is fixed at 30 dB. Two scenarios are used. In Figures 1–2 and in Figures 3–8, uniform and Gaussian noise are used, respectively.

In Figures 2, 4, 6, and 8, the correlated data is obtained by passing a unit-variance i.i.d. Gaussian data through a first-order auto-regressive model with transfer function $\frac{\sqrt{1-a^2}}{1-az^{-1}}$ and $a = 0.8$.

As depicted from Figures 1–4, the performance of the LMS algorithm is found to be superior when compared with that of the SSLMS algorithm for complex-valued data. Figures 5–6 show the steady-state performance of the sign–sign LMS algorithm using white and correlated Gaussian regressors, respectively. Figures 7–8 show the tracking performance of the sign–sign LMS algorithm using white and correlated Gaussian regressors, respectively. As can be seen from Figures 5–8, the simulation and analytical results are in relatively better agreement for white Gaussian regressors than correlated Gaussian regressors for complex-valued data.

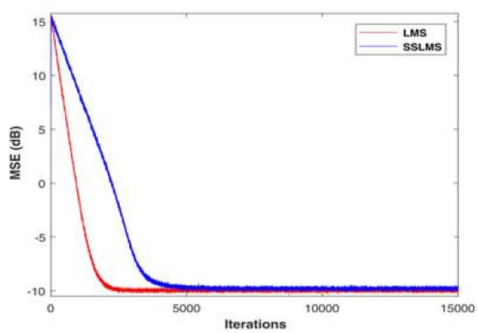

Fig. 1. Comparison of the SSLMS and LMS adaptive filters in a uniform noise for white Gaussian regressors.

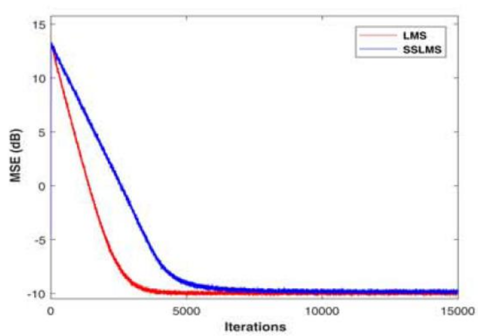

Fig. 2. Comparison of the SSLMS and LMS adaptive filters in a uniform noise for correlated Gaussian regressors.

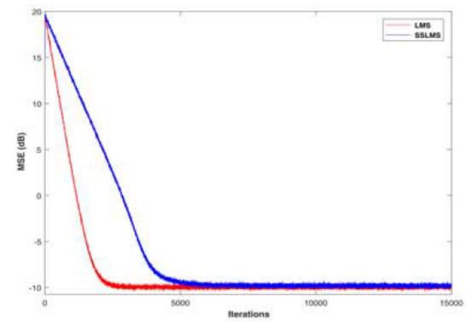

Fig. 3. Comparison of the SSLMS and LMS adaptive filters in a Gaussian noise for white Gaussian regressors.

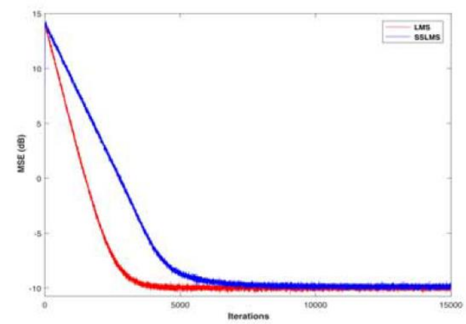

Fig. 4. Comparison of the SSLMS and LMS adaptive filters in a Gaussian noise for correlated Gaussian regressors.

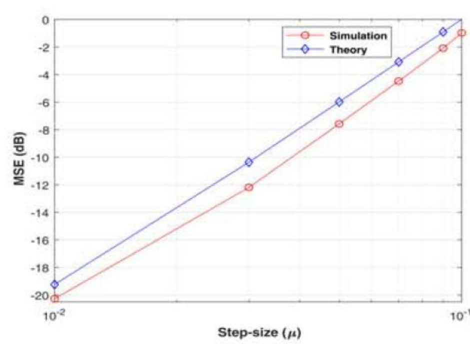

Fig. 5. Theoretical and simulated steady-state MSE of the sign–sign LMS algorithm using white Gaussian regressors in an AWGN environment.

V. CONCLUSIONS

This work reports terms for the steady-state MSE, the optimum step-size, and the corresponding minimum value of the tracking MSE of the SSLMS algorithm for complex-valued data. It is shown that the convergence behaviour of the LMS algorithm is superior to that of the SSLMS algorithm. Finally, a good agreement between the simulation and the theoretical results for white Gaussian regressors than correlated Gaussian regressors is obtained.

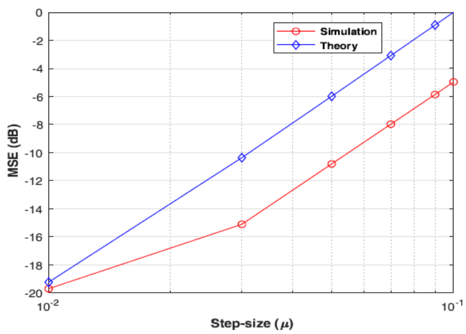

Fig. 6. Theoretical and simulated steady-state MSE of the sign–sign LMS algorithm using correlated Gaussian regressors in an AWGN environment.

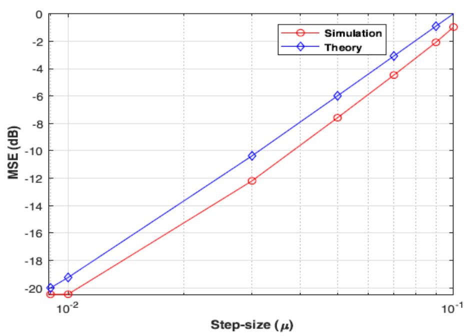

Fig. 7. Theoretical and simulated tracking MSE of the sign–sign LMS algorithm using white Gaussian regressors in an AWGN environment.

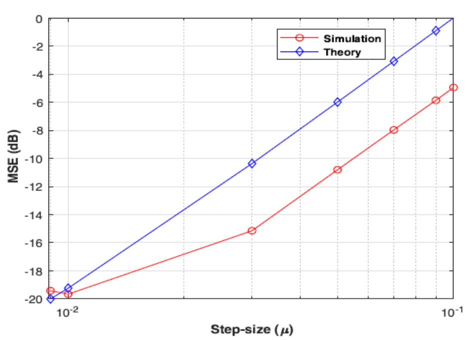

Fig. 8. Theoretical and simulated tracking MSE of the sign–sign LMS algorithm using correlated Gaussian regressors in an AWGN environment.

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Insights into the convergence and steady-state behaviors of the SLMF and its variants

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***Abstract*—In this paper, we provide some insights into the convergence and steady-state behaviors of the sign-error least mean fourth (SLMF), sign-sign least mean fourth (SSLMF), normalized sign-error least mean fourth (NSLMF), and normalized sign-sign least mean fourth (NSSLMF) algorithms for both cases of real- and complex-valued data. Moreover, we also report the equivalence algorithms of the block-based normalized sign-error least mean fourth (BBNSLMF) and block-based normalized sign-sign least mean fourth (BBNSSLMF) algorithms. Finally, simulations are conducted for both cases of real- and complex-valued data to provide us with more insights into the performance of the SLMF, SSLMF, NSLMF, and NSSLMF algorithms.**

***Index Terms*—Convergence, Steady-state, SLMF, NSLMF, BB-NSLMF, SSLMF, NSSLMF, BBNSSLMF.**

I. Introduction

In [1], eight adaptive noise cancelers were proposed for electroencephalogram (EEG) signal enhancement in remote health monitoring applications. These eight adaptive noise cancelers were implemented using the variants of least mean fourth (LMF) algorithm [2], such as the normalized least mean fourth (NLMF) [3]–[4], normalized sign regressor least mean fourth (NSRLMF) [5]–[6], normalized sign-error least mean fourth (NSLMF), normalized sign-sign least mean fourth (NSSLMF), block-based normalized least mean fourth (BBNLMF), block-based normalized sign regressor least mean fourth (BBNSRLMF), block-based normalized sign-error least mean fourth (BBNSLMF), and block-based normalized sign-sign least mean fourth (BBNSSLMF) algorithms [1]. Four of these adaptive noise cancelers implemented using the NSLMF, NSSLMF, BBNSLMF, and BBNSSLMF algorithms [1] are the motivation behind this work.

The organization of this paper is as follows. The sign-error least mean fourth (SLMF) algorithm, which is the basis for developing the NSLMF and BBNSLMF algorithms [1] is described in Section II. The sign-sign least mean fourth (SSLMF) algorithm, which is the basis for developing the NSSLMF and BBNSSLMF algorithms [1] is described in Section III. The NSLMF and NSSLMF algorithms [1] are described in Sections IV and V, respectively. Simulation results are discussed in Section VI. Finally, the paper is concluded in Section VII.

II. The SLMF Algorithm

The SLMF algorithm [1] is based on clipping of the estimation error signal, whose weight update equation for real-valued data is given by

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\, \mathbf{u}_i^{\mathrm{T}} \mathrm{sign}[e_i^3], \quad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ is the updated weight vector, μ is the step-size, $\mathbf{u}_i$ is the regressor vector, the estimation error signal e_i is given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}, \tag{2}$$

where d_i is the desired value, and

$$\mathrm{sign}[x] = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases} \tag{3}$$

We know that,

$$\mathrm{sign}[x^3] = \mathrm{sign}[x]. \tag{4}$$

Therefore, the SLMF algorithm in (1) boils down to the sign-error least mean square (SLMS) algorithm as shown below [7]–[8]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\, \mathbf{u}_i^{\mathrm{T}} \mathrm{sign}[e_i], \quad i \geq 0. \tag{5}$$

In Table I we give some examples for the validity of equation (4).

TABLE I
Some Examples For Equation (4).

x	$\mathrm{sign}[x]$	$\mathrm{sign}[x^3]$
0	0	0
0.1	1	1
−0.1	−1	−1

For complex-valued data [9], the weight update equation of the SLMF algorithm takes the form

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\, \mathbf{u}_i^{*} \mathrm{csgn}[e_i|e_i|^2], \quad i \geq 0, \tag{6}$$

where

$$\mathrm{csgn}[x] = \begin{cases} -1, & \text{if } \Re[x] < 0 \text{ or} \\ & (\Re[x] = 0 \text{ and } \Im[x] < 0), \\ 0, & \text{if } \Re[x] = \Im[x] = 0, \\ 1, & \text{if } \Re[x] > 0 \text{ or} \\ & (\Re[x] = 0 \text{ and } \Im[x] > 0). \end{cases} \tag{7}$$

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We know that $|e_i|^2$ is a positive number. The multiplication of $|e_i|^2$ with e_i yields a scaled version of the estimation error signal while preserving the signs of its real and imaginary parts. Therefore, the complex SLMF algorithm in (6) reduces to the complex SLMS algorithm as shown below [7]–[8]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathbf{u}_i^* \text{csgn}[e_i], \qquad i \geq 0. \tag{8}$$

In Table II we present some examples for the following equation:

$$\text{csgn}[x|x|^2] = \text{csgn}[x]. \tag{9}$$

TABLE II
SOME EXAMPLES FOR EQUATION (9).

x	csgn$[x]$	csgn$[x\|x\|^2]$
0	0	0
$0.1 + 0.1i$	1	1
$0.1 - 0.1i$	1	1
$-0.1 + 0.1i$	-1	-1
$-0.1 - 0.1i$	-1	-1

III. THE SSLMF ALGORITHM

The SSLMF algorithm [1] is obtained by clipping both the regressor vector and the estimation error signal as shown below for real-valued data:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \text{sign}[\mathbf{u}_i]^{\text{T}} \text{sign}[e_i^3], \qquad i \geq 0. \tag{10}$$

Putting equation (4) into (10) we observe that the SSLMF algorithm boils down to the sign-sign least mean square (SSLMS) algorithm as shown below [8], [10]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \text{sign}[\mathbf{u}_i]^{\text{T}} \text{sign}[e_i], \qquad i \geq 0. \tag{11}$$

For complex-valued data, the weight update rule of the SSLMF algorithm takes the following form:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \text{csgn}[\mathbf{u}_i]^* \text{csgn}[e_i|e_i|^2], \qquad i \geq 0, \tag{12}$$

which boils down to the complex SSLMS algorithm as described below [8], [10]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \text{csgn}[\mathbf{u}_i]^* \text{csgn}[e_i], \qquad i \geq 0. \tag{13}$$

The MATLAB implementation of the complex SSLMS algorithm is as follows:

$$\begin{aligned}\mathbf{w}_i \;=\;& \mathbf{w}_{i-1} + \mu\ \{\text{sign}[\text{real}(\mathbf{u}_i')] + \text{sign}[\text{imag}(\mathbf{u}_i')]i\} \\ & \times\{\text{sign}[\text{real}(e_i)] + \text{sign}[\text{imag}(e_i)]i\}. \end{aligned} \tag{14}$$

Also, the MATLAB implementation for the complex SSLMF algorithm is

$$\begin{aligned}&\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \{\text{sign}[\text{real}(\mathbf{u}_i')] + \text{sign}[\text{imag}(\mathbf{u}_i')]i\} \\ &\times\{\text{sign}[\text{real}(e_i)\text{abs}(e_i)^2] + \text{sign}[\text{imag}(e_i)\text{abs}(e_i)^2]i\}. \end{aligned} \tag{15}$$

IV. THE NSLMF ALGORITHM

The weight update recursion of the NSLMF algorithm [1] for real-valued data is given by:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\ \mathbf{u}_i^{\text{T}} \text{sign}[e_i^3], \qquad i \geq 0, \tag{16}$$

$$\mu_i = \frac{\mu}{\epsilon + \text{sign}[\mathbf{u}_i]\text{sign}[\mathbf{u}_i]^{\text{T}}}, \tag{17}$$

where μ_i is the normalized step-size and ϵ is an extremely small positive constant to avoid division by zero whenever the regressor is zero. When the data is complex-valued, the weight update recursion of the NSLMF algorithm is governed by

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\ \mathbf{u}_i^* \text{csgn}[e_i|e_i|^2], \qquad i \geq 0, \tag{18}$$

$$\mu_i = \frac{\mu}{\epsilon + \text{csgn}[\mathbf{u}_i]\text{csgn}[\mathbf{u}_i]^*}. \tag{19}$$

Based on the discussion in Section II, the NSLMF algorithm in (16) and (18) gets reduced to the normalized sign-error least mean square (NSLMS) algorithm [11] as described by the equations (20) and (21), respectively, for real- and complex-valued data cases.

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\ \mathbf{u}_i^{\text{T}} \text{sign}[e_i], \qquad i \geq 0. \tag{20}$$

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\ \mathbf{u}_i^* \text{csgn}[e_i], \qquad i \geq 0. \tag{21}$$

V. THE NSSLMF ALGORITHM

The weight update recursions of the NSSLMF algorithm [1] for real- and complex-valued data cases are given by the following two equations, respectively.

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\ \text{sign}[\mathbf{u}_i]^{\text{T}} \text{sign}[e_i^3], \qquad i \geq 0. \tag{22}$$

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\ \text{csgn}[\mathbf{u}_i]^* \text{csgn}[e_i|e_i|^2], \qquad i \geq 0. \tag{23}$$

Similarly, in this case too, the NSSLMF algorithm in (22) and (23) gets reduced to the normalized sign-sign least mean square (NSSLMS) algorithm [12]–[13] as described by the equations (24) and (25), respectively, for real- and complex-valued data cases.

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\ \text{sign}[\mathbf{u}_i]^{\text{T}} \text{sign}[e_i], \qquad i \geq 0. \tag{24}$$

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu_i\ \text{csgn}[\mathbf{u}_i]^* \text{csgn}[e_i], \qquad i \geq 0. \tag{25}$$

The normalization term used in [1] for the NSLMF and NSSLMF algorithms is slightly different than the one we have used in this work. No matter how one chooses to normalize, the NSLMF and NSSLMF algorithms will eventually boil down to the NSLMS and NSSLMS algorithms, respectively.

In [1], the update equations of the SLMF, SSLMF, NSLMF, and NSSLMF algorithms are given for the case of real-valued data. For the sake of completeness, in this paper we have discussed the SLMF, SSLMF, NSLMF, and NSSLMF algorithms for both cases of real- and complex-valued data.

Finally, Table III reports the equivalence algorithms of the block-based algorithms proposed in [1].

TABLE III
BLOCK-BASED ALGORITHMS AND THEIR EQUIVALENTS.

Block-based Algorithm	Equivalent Algorithm
BBNSLMF	BBNSLMS
BBNSSLMF	BBNSSLMS

VI. SIMULATION RESULTS

In all the simulations, the adaptive weight vector length is fixed at $M = 10$ and an additive white Gaussian noise (AWGN) environment is considered at a signal-to-noise ratio (SNR) of 10 dB. The value of ϵ is fixed at 10^{-6} for all the normalized algorithms.

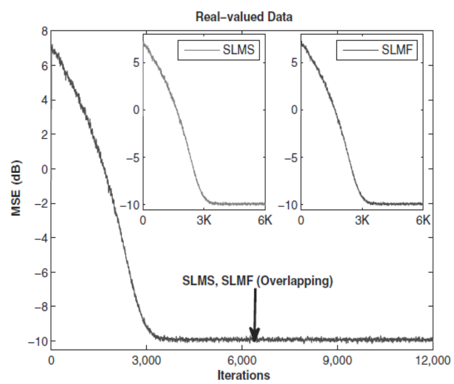

Fig. 1. Convergence behavior of the SLMS and SLMF algorithms.

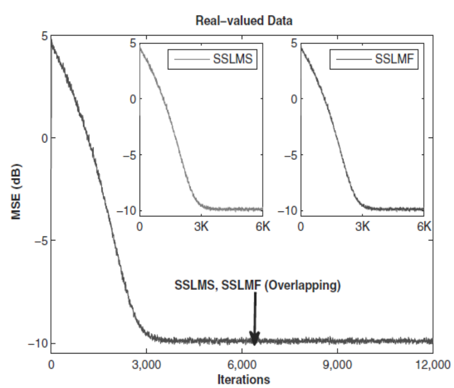

Fig. 2. Convergence behavior of the SSLMS and SSLMF algorithms.

Figures 1–4 demonstrate the convergence performance of the SLMF, SSLMF, NSLMF, and NSSLMF algorithms for the case of real-valued data, and while Figures 5–8 demonstrate their convergence performance for the case of complex-valued data. As can be seen from Figures 1–8, the SLMF, SSLMF, NSLMF, and NSSLMF algorithms converge at exactly the same rate and have the same mean-square error (MSE) performance as the SLMS, SSLMS, NSLMS, and NSSLMS algorithms, respectively.

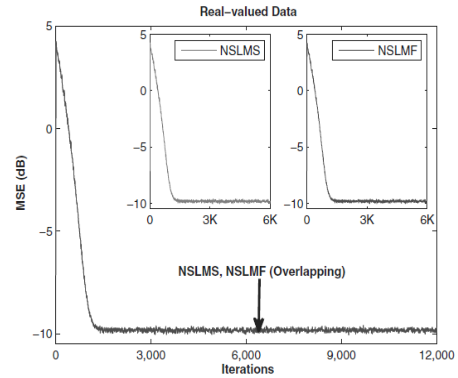

Fig. 3. Convergence behavior of the NSLMS and NSLMF algorithms.

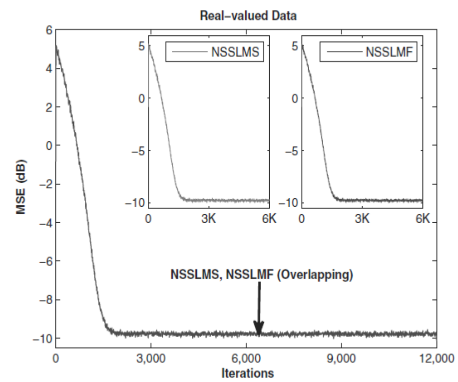

Fig. 4. Convergence behavior of the NSSLMS and NSSLMF algorithms.

As depicted in Figures 1–8, the NSLMF and NSSLMF algorithms outperform the SLMF and SSLMF algorithms, respectively. In Figures 1, 3, 5, and 7 it is observed that the SLMF and NSLMF algorithms converge faster for real-valued data than complex-valued data. Finally, it is noted in Figures 2, 4, 6, and 8 that the convergence performance of the SSLMF and NSSLMF algorithms is superior for complex-valued data than real-valued data.

VII. CONCLUSIONS

In this work, we have shown that the convergence and MSE performance of the SLMF, SSLMF, NSLMF, and NSSLMF algorithms are exactly the same as those of the SLMS, SSLMS, NSLMS, and NSSLMS algorithms, respectively, for both the real- and complex-valued data cases. As was expected, the NSLMF and NSSLMF algorithms converge faster than the SLMF and SSLMF algorithms, respectively, for both the real- and complex-valued data cases. We observed that the convergence behavior of the SLMF and NSLMF algorithms is better when the data is real-valued than complex-valued. It is interesting to note that the SSLMF and NSSLMF algorithms converge faster for complex-valued data than real-valued data.

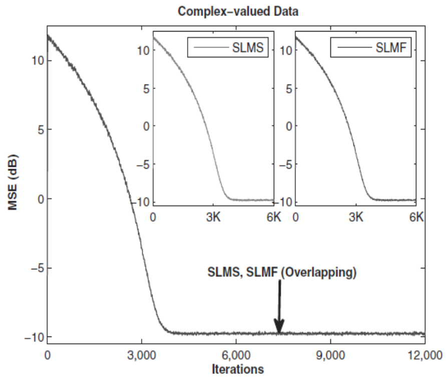

Fig. 5. Convergence behavior of the complex SLMS and complex SLMF algorithms.

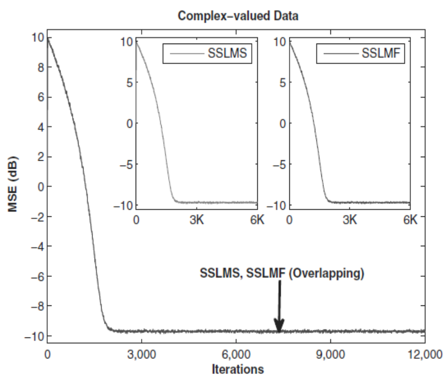

Fig. 6. Convergence behavior of the complex SSLMS and complex SSLMF algorithms.

Acknowledgment

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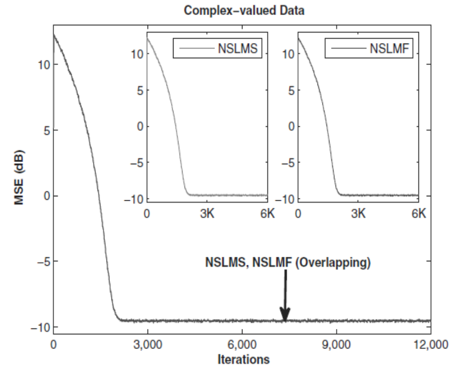

Fig. 7. Convergence behavior of the complex NSLMS and complex NSLMF algorithms.

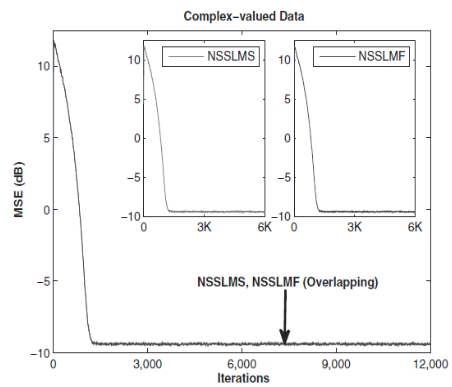

Fig. 8. Convergence behavior of the complex NSSLMS and complex NSSLMF algorithms.

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IEEE SENSORS JOURNAL, VOL. 15, NO. 9, SEPTEMBER 2015 5349

Comments and Corrections

Comments on "Efficient Signal Conditioning Techniques for Brain Activity in Remote Health Monitoring Network"

Mohammed Mujahid Ulla Faiz

***Abstract*— The main purpose of this paper is to remove misconceptions among biomedical signal processing researchers concerning the implementation of adaptive noise cancelers using the sign-error least mean fourth (SELMF), sign-sign least mean fourth (SSLMF), and their variant algorithms.**

***Index Terms*— Adaptive noise cancelers, SELMF, SSLMF.**

I. Introduction

In [1], the authors proposed several adaptive noise cancelers for electroencephalogram (EEG) signal enhancement in remote health monitoring applications. Four of these adaptive noise cancelers proposed in [1] are implemented using the normalized sign-error least mean fourth (NSELMF), block-based normalized sign-error least mean fourth (BBNSELMF), normalized sign-sign least mean fourth (NSSLMF), and block-based normalized sign-sign least mean fourth (BBNSSLMF) algorithms. The former two algorithms are the variants of the sign-error least mean fourth (SELMF) algorithm and the latter two algorithms are the variants of the sign-sign least mean fourth (SSLMF) algorithm.

II. The SELMF and SSLMF Algorithms

The SELMF algorithm is obtained from the conventional least mean fourth (LMF) algorithm [2] by replacing the estimation error signal e_i by its sign. The weight vector update equation of the SELMF algorithm is given by

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathbf{u}_i^{\mathrm{T}} \mathrm{sign}[e_i^3], \quad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ is the updated weight column vector, μ is the step-size, $\mathbf{u}_i$ is the regressor row vector, the estimation error signal e_i is given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}, \tag{2}$$

where d_i is the desired value, and

$$\mathrm{sign}[x] = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases} \tag{3}$$

We know that,

$$\mathrm{sign}[x^3] = \mathrm{sign}[x]. \tag{4}$$

Therefore, the SELMF algorithm in (1) reduces to the sign-error least mean square (SELMS) algorithm as given below [3], [4]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathbf{u}_i^{\mathrm{T}} \mathrm{sign}[e_i], \quad i \geq 0. \tag{5}$$

The SSLMF algorithm is based on clipping of both the regressor vector and the estimation error signal, whose weight update rule is given by

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} \mathrm{sign}[e_i^3], \quad i \geq 0. \tag{6}$$

Substituting equation (4) in (6) we find that the SSLMF algorithm reduces to the sign-sign least mean square (SSLMS) algorithm as given below [4], [5]:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu\ \mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} \mathrm{sign}[e_i], \quad i \geq 0. \tag{7}$$

In [1], the SELMF and SSLMF algorithms are wrongly cited to be mentioned in our paper on the analysis of sign regressor least mean fourth (SRLMF) algorithm [6]. There is absolutely no mention of the SELMF and SSLMF algorithms in [6].

Note that the SRLMF algorithm [6] is referred to as the sign clipped least mean fourth (SCLMF) algorithm in [1]. Furthermore, the variants of the SRLMF algorithm [6] such as the normalized sign regressor least mean fourth (NSRLMF) [7], [8] and block-based normalized sign regressor least mean fourth (BBNSRLMF) algorithms are referred to as the normalized clipped least mean fourth (NCLMF) and block-based normalized clipped least mean fourth (BBNCLMF) algorithms, respectively, in [1].

III. Conclusions

The adaptive noise cancelers implemented using the SELMF, SSLMF, and their variant algorithms in [1] for EEG signal enhancement would deliver exactly the same performance in terms of the convergence rate and mean-square error (MSE) as compared to the adaptive noise cancelers implemented using the SELMS [3], [4], SSLMS [4], [5], and their corresponding variant algorithms [9], [10], respectively, provided the parameter settings and EEG data used are same. Thus, there is no point in conducting research on the SELMF, SSLMF, and their variant algorithms as they are exactly identical to the SELMS [3], [4], SSLMS [4], [5], and their corresponding variant algorithms [9], [10], respectively. Note that only the SRLMF [6] and its variant algorithms [7], [8] are distinct from the sign regressor least mean square (SRLMS) [11] and its corresponding variant algorithms [12], [13], respectively.

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The $\epsilon-$Normalized Sign Regressor Least Mean Square (NSRLMS) Adaptive Algorithm

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***Abstract*—In this paper, expressions are derived for the steady-state and tracking excess-mean-square error (EMSE) of the $\epsilon-$normalized sign regressor least mean square (NSRLMS) adaptive algorithm. Finally, it is shown that simulations performed for both the cases of white and correlated Gaussian regressors substantiate very well the theory developed.**

I. Introduction

The sign based variants of the least mean square (LMS) algorithm [1] were introduced due to the simplicity of their implementation. The sign regressor algorithm (SRA) is one such variant of the LMS algorithm, which is based on clipping of the input data [2]. However, these sign based algorithms result in a performance loss when compared with the LMS algorithm [3].

In [4], it is shown that the normalized least mean square (NLMS) algorithm converges faster than the LMS algorithm. A sign version of the NLMS algorithm, the normalized sign regressor least mean square (NSRLMS) algorithm or simply the normalized sign regressor algorithm (NSRA) as it is more commonly known combines the advantages of the NLMS and SRA algorithms. Theoretical studies of the NSRLMS algorithm can be found in [5]–[6]. In [7], the NSRLMS was tested in an adaptive noise cancellation scenario in order to remove noise from the electrocardiogram (ECG) signal. In our work, expressions are evaluated for the steady-state excess-mean-square error (EMSE) of the $\epsilon-$NSRLMS algorithm in a stationary environment. Also, expressions for the tracking EMSE in a nonstationary environment are presented. The framework used in our analysis relies on energy conservation arguments [8]. From the simulation results it is shown that the theoretical and simulated results are in very good agreement.

The organization of the paper is as follows. In Section II, the $\epsilon-$NSRLMS algorithm is described. The mean-square analysis and the tracking analysis of the $\epsilon-$NSRLMS algorithm is presented in Sections III and IV, respectively. Finally, simulation results are discussed in Section V and Section VI concludes the paper.

II. The $\epsilon-$NSRLMS Algorithm

Consider a zero-mean random variable d with realizations $\{d(0), d(1), \ldots\}$, and a zero-mean random row vector $\mathbf{u}$ with realizations $\{\mathbf{u}_0, \mathbf{u}_1, \ldots\}$. The optimal weight vector $\mathbf{w}^o$ that solves:

$$\min_{\mathbf{w}} \mathrm{E}|d - \mathbf{u}\mathbf{w}|^2, \tag{1}$$

can be approximated iteratively via the recursion (the $\epsilon-$NSRLMS algorithm)

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}} \operatorname{csgn}[\mathbf{u}_i]^* e_i, \quad i \geq 0, \tag{2}$$

where $\mathbf{w}_i$ (column vector) is the updated weight vector at time i, μ is the step-size, ϵ is a small positive constant to avoid division by zero when the regressor is zero, H[.] is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$, and e_i denotes the estimation error signal given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}. \tag{3}$$

III. Mean-Square Analysis of the $\epsilon-$NSRLMS Algorithm

We shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the stationary data model [8]:

- A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that $d_i = \mathbf{u}_i\mathbf{w}^o + v_i$.
- A.2 The noise sequence v_i is independent and identically distributed (i.i.d.) circular with variance $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ and is independent of $\mathbf{u}_j$ for all i, j.
- A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.
- A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^*\mathbf{u}_i] > \mathbf{0}$.

For the adaptive filter of the form in (2), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [8]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}} |\mathrm{g}[e_i]|^2\right] = 2\mathrm{Re}\left[\mathrm{E}\left[e^*_{a_i}\mathrm{g}[e_i]\right]\right], \quad \text{as } i \to \infty, \tag{4}$$

where

$$\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}] = \mathrm{E}[\mathrm{Re}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*]], \tag{5}$$

$$e_i = e_{a_i} + v_i, \tag{6}$$

with g[.] denoting some function of e_i, and $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error. Then $\mathrm{g}[e_i]$ for the $\epsilon-$NSRLMS algorithm becomes

$$\mathrm{g}[e_i] = \frac{e_{a_i} + v_i}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}. \tag{7}$$

By using the fact that e_{a_i} and v_i are independent, we reach at the following expression for the term $\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]$:

$$\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right] = \mathrm{E}\left[\frac{|e_{a_i}|^2}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{8}$$

To evaluate the term $\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right]$, we start by noting that

$$|\mathrm{g}[e_i]|^2 = \frac{1}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\left[|e_{a_i}|^2 + |v_i|^2 + e_{a_i}^* v_i + e_{a_i} v_i^*\right]. \tag{9}$$

If we multiply $|\mathrm{g}[e_i]|^2$ by $||\mathbf{u}_i||_{\mathrm{H}}^2$ from the left, and use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, we obtain

$$\begin{aligned}\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right] &= \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2 |e_{a_i}|^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right] + \sigma_v^2 \\ &\times \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]. \end{aligned} \tag{10}$$

Substituting (8) and (10) into (4) we get

$$\begin{aligned}2\mathrm{Re}\left[\mathrm{E}\left[\frac{|e_{a_i}|^2}{\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\right] &= \mu \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2 |e_{a_i}|^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right] + \mu\sigma_v^2 \\ &\times \mathrm{E}\left[\frac{||\mathbf{u}_i||_{\mathrm{H}}^2}{(\epsilon + ||\mathbf{u}_i||_{\mathrm{H}}^2)^2}\right]. \end{aligned} \tag{11}$$

By using the assumption $\epsilon \approx 0$ in (11), we obtain

$$2\mathrm{Re}\left[\mathrm{E}\left[\frac{|e_{a_i}|^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]\right] = \mu \mathrm{E}\left[\frac{|e_{a_i}|^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] + \mu\sigma_v^2 \mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{12}$$

Now, let us use the following steady-state approximation:

$$\mathrm{E}\left[\frac{|e_{a_i}|^2}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] \approx \frac{\mathrm{E}[|e_{a_i}|^2]}{\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]}. \tag{13}$$

From Price's theorem [9] we have

$$\mathrm{E}\left[\mathrm{Re}[x^* \mathrm{csgn}(y)]\right] = \sqrt{\frac{2}{\pi}} \frac{\sqrt{2}}{\sigma_y} \mathrm{E}\left[\mathrm{Re}[x^* y]\right], \tag{14}$$

where x and y denote two complex-valued jointly-Gaussian random variables. Therefore,

$$\begin{aligned}\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] &= \mathrm{E}[\mathrm{Re}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*]], \\ &= \mathrm{E}\left[\mathrm{Re}[\mathbf{u}_i \mathrm{csgn}[\mathbf{u}_i]^*]\right], \\ &= \frac{4\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}}. \end{aligned} \tag{15}$$

Substituting (13) and (15) into (12) we get

$$\frac{\sqrt{\pi\sigma_u^2}\mathrm{E}[|e_{a_i}|^2]}{2\mathrm{Tr}(\mathbf{R})} = \frac{\mu\sqrt{\pi\sigma_u^2}\mathrm{E}[|e_{a_i}|^2]}{4\mathrm{Tr}(\mathbf{R})} + \mu\sigma_v^2 \mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{16}$$

Therefore, the steady-state EMSE $\zeta = \mathrm{E}[|e_{a_i}|^2]$ of the $\epsilon-$NSRLMS algorithm can be shown to be

$$\zeta = \frac{4\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{(2-\mu)\sqrt{\pi\sigma_u^2}} \mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right]. \tag{17}$$

IV. Tracking Analysis of the $\epsilon-$NSRLMS Algorithm

Here, we assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the nonstationary data model [8]:

- A.5 There exists a vector $\mathbf{w}_i^o$ such that $d_i = \mathbf{u}_i \mathbf{w}_i^o + v_i$.
- A.6 The weight vector varies according to the random-walk model $\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i$, and the sequence $\mathbf{q}_i$ is i.i.d. with covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.
- A.7 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

In this case, the following variance relation holds [8]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]\right], \quad \text{as } i \to \infty. \tag{18}$$

Tracking results can be obtained by inspection from the mean-square results as there are only minor differences. Therefore, by substituting (8) and (10) into (18) we get

$$\mu\sigma_v^2 \mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = \frac{(2-\mu)\sqrt{\pi\sigma_u^2}\mathrm{E}[|e_{a_i}|^2]}{4\mathrm{Tr}(\mathbf{R})}. \tag{19}$$

Therefore, the tracking EMSE ζ of the $\epsilon-$NSRLMS algorithm is given by

$$\zeta = \frac{4\mathrm{Tr}(\mathbf{R})}{(2-\mu)\sqrt{\pi\sigma_u^2}}\left[\mu\sigma_v^2 \mathrm{E}\left[\frac{1}{||\mathbf{u}_i||_{\mathrm{H}}^2}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\right]. \tag{20}$$

Moreover, in both cases, the expression for the mean-square error (MSE) of the $\epsilon-$NSRLMS algorithm is obtained by

$$\mathrm{E}\left[|e_i|^2\right] = \zeta + \sigma_v^2. \tag{21}$$

V. Simulation Results

First, in order to validate the theoretical findings extensive simulations are carried out for different scenarios. Figures 1-2 are for the case of the steady-state MSE of a 10-tap $\epsilon-$NSRLMS filter in a stationary environment and Figures 3-4 are for the case of the tracking MSE in a nonstationary environment. In all of these figures the MSE is plotted as a function of the step-size μ for a signal to noise ratio (SNR) of 30 dB and the value of ϵ is set to 10^{-6}. Moreover, all the simulations reported in this work use complex-valued signals.

In the case of Figures 1 and 3, the regressors, with shift structure, are generated by feeding a unit-variance white process into a tapped delay line. However, in Figures 2 and 4, the regressors, with shift structure, are generated by passing correlated data into a tapped delay line. Here, the correlated data is obtained by passing a unit-variance i.i.d. Gaussian data through a first-order auto-regressive model with transfer function $\frac{\sqrt{1-a^2}}{(1-az^{-1})}$ and $a = 0.8$. As can be seen from Figures 1-2, the simulation results match very well the theoretical result (17), which is the steady-state EMSE of the $\epsilon-$NSRLMS algorithm.

Finally, to further validate the theoretical results in a tracking scenario, Figures 3-4 depict this behavior. Here, the random-walk channel behaves according to

$$\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i, \tag{22}$$

where $\mathbf{q}_i$ is a Gaussian sequence with zero mean and variance $\sigma_q^2 = 10^{-9}$. As observed from Figures 3-4, the simulation results corroborate closely the theoretical result (20), which is the tracking EMSE of the $\epsilon-$NSRLMS algorithm.

VI. Conclusions

The mean-square analysis and the tracking analysis of the $\epsilon-$NSRLMS algorithm is carried out. Moreover, simulations performed are found to closely corroborate with the analytical results.

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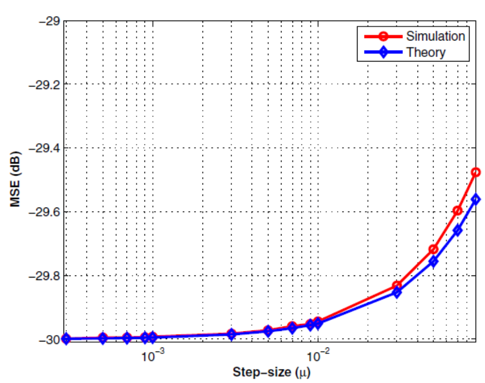

Fig. 1. Theoretical and simulated steady-state MSE of the $\epsilon-$NSRLMS algorithm using white Gaussian regressors.

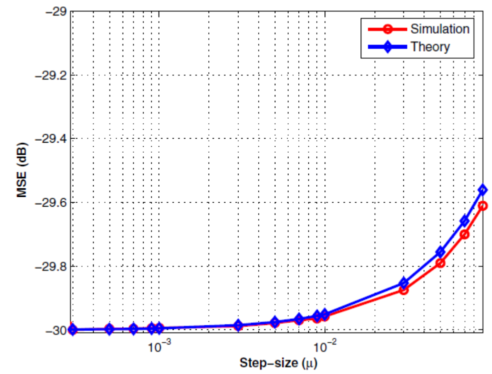

Fig. 2. Theoretical and simulated steady-state MSE of the $\epsilon-$NSRLMS algorithm using correlated Gaussian regressors.

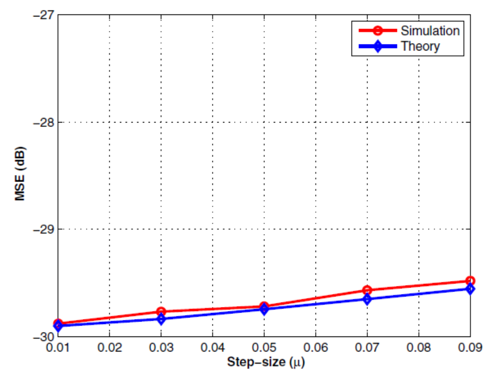

Fig. 3. Theoretical and simulated tracking MSE of the $\epsilon-$NSRLMS algorithm using white Gaussian regressors.

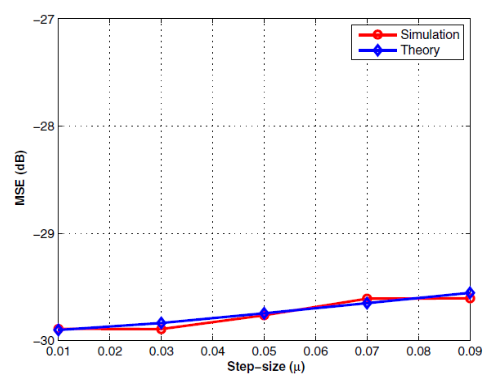

Fig. 4. Theoretical and simulated tracking MSE of the $\epsilon-$NSRLMS algorithm using correlated Gaussian regressors.

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A Note on NSRLMS, NSRLMF, and NSRLMMN Adaptive Algorithms

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***Abstract*—In this paper, we compare the expressions for the steady-state mean-square error (MSE), the optimum step-size, and the corresponding minimum tracking MSE of the normalized sign regressor least mean square (NSRLMS), the normalized sign regressor least mean fourth (NSRLMF), and the normalized sign regressor least mean mixed-norm (NSRLMMN) algorithms for the case of real-valued data. The expressions for the steady-state MSE, the optimum step-size, and the corresponding minimum tracking MSE of the NSRLMF and NSRLMMN algorithms based on energy conservation relation approach are available in the literature for the case of real-valued data. Thus, in order to compare these three algorithms, we have derived the expressions for the steady-state MSE, the optimum step-size, and the corresponding minimum tracking MSE of the NSRLMS algorithm based on energy conservation relation approach for the case of real-valued data. Finally, simulation results to substantiate the analytical results of the NSRLMS algorithm are also presented for the case of real-valued data.**

***Index Terms*—NSRLMS, NSRLMF, NSRLMMN, steady-state, tracking.**

I. Introduction

In [1]– [2], we had proposed the normalized sign regressor least mean mixed-norm (NSRLMMN) algorithm, which is based on clipping of the regressor vector $\mathbf{u}_i$. The NSRLMMN algorithm updates its weight vector $\mathbf{w}_{i(\mathrm{NSRLMMN})}$ according to the following recursive rule:

$$\begin{aligned}\mathbf{w}_{i(\mathrm{NSRLMMN})} &= \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i[\delta \\ &\quad +\bar{\delta}e_i^2], \quad i \geq 0,\end{aligned} \tag{1}$$

where μ is the step-size, ϵ is an extremely small positive constant to avoid division by zero when the regressor is zero, δ is the mixing parameter, $\bar{\delta} = 1-\delta$, e_i is the estimation error, $||\mathbf{u}_i||^2_{\mathrm{H}} = \mathbf{u}_i\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, and $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined for filter length M as

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i_M}|}\right\}. \tag{2}$$

The NSRLMMN algorithm reduces to normalized sign regressor least mean fourth (NSRLMF) and normalized sign regressor least mean square (NSRLMS) algorithms when δ takes the value 0 and 1, respectively. The weight update equations of the NSRLMF [3]– [4] and NSRLMS [5]– [8] algorithms are thus given by (3) and (4), respectively:

$$\mathbf{w}_{i(\mathrm{NSRLMF})} = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \tag{3}$$

$$\mathbf{w}_{i(\mathrm{NSRLMS})} = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i. \tag{4}$$

Adaptive filters equipped with the NSRLMS algorithm are extensively used for the processing and analysis of electrocardiogram (ECG) signals [7]– [8]. To the best of the authors' knowledge, the steady-state and tracking analysis of the NSRLMS algorithm based on energy conservation relation approach [9]– [10] are not available in the literature. Therefore, this work reports the findings of the steady-state and tracking analysis of the NSRLMS algorithm based on energy conservation relation approach.

II. Steady-State Analysis

In [1], the expression for the steady-state mean-square error (MSE), $\varphi_{\mathrm{NSRLMMN}} = \mathrm{E}\left[e_i^2\right]$, of the NSRLMMN algorithm is given by

$$\begin{aligned}\varphi &= \frac{\mu(\delta^2\sigma_v^2 + \bar{\delta}^2\xi_v^6 + 2\delta\bar{\delta}\xi_v^4)}{2(\delta + 3\bar{\delta}\sigma_v^2) - \mu(\delta^2 + 15\bar{\delta}^2\xi_v^4 + 12\delta\bar{\delta}\sigma_v^2)} \\ &\quad +\sigma_v^2,\end{aligned} \tag{5}$$

where $\sigma_v^2 = \mathrm{E}[v_i^2]$, $\xi_v^4 = \mathrm{E}[v_i^4]$, and $\xi_v^6 = \mathrm{E}[v_i^6]$ denote the second, fourth, and sixth-order moments of the noise sequence, v_i, respectively.

The steady-state MSE expressions of the NSRLMF [3] and NSRLMS algorithms can be easily obtained by substituting δ is equal to 0 and 1 in (5), respectively, as shown below:

$$\varphi_{\text{NSRLMF}} = \frac{\mu \xi_v^6}{6\sigma_v^2 - 15\mu\xi_v^4} + \sigma_v^2, \quad (6)$$

$$\varphi_{\text{NSRLMS}} = \frac{\mu \sigma_v^2}{2-\mu} + \sigma_v^2. \quad (7)$$

Note that the steady-state MSE expression of the NSRLMF algorithm in (6) is exactly the same as the one derived by us in [3].

To carry out the steady-state analysis of the NSRLMS algorithm we shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the stationary data model [1], [3], [9]– [11]:

- A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that the desired sequence $d_i = \mathbf{u}_i\mathbf{w}^o + v_i$.
- A.2 The noise sequence v_i is independent and identically distributed (i.i.d.) with variance σ_v^2 and is independent of $\mathbf{u}_j$ for all i, j.
- A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.
- A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i] > \mathbf{0}$.

For the adaptive filter of the form in (4), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [1], [3], [9]– [11]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}} \mathrm{g}^2[e_i]\right] = 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \to \infty, \quad (8)$$

where $e_i = e_{a_i} + v_i$, $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error, and $\mathrm{g}[e_i]$ for the NSRLMS algorithm can be written as

$$\mathrm{g}[e_i] = \frac{e_{a_i} + v_i}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}. \quad (9)$$

By using the fact that e_{a_i} and v_i are independent, we reach at the following expression for the term $\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right]$:

$$\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right] = \mathrm{E}\left[\frac{e_{a_i}^2}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\right]. \quad (10)$$

To evaluate the term $\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\mathrm{g}^2[e_i]\right]$, we start by noting that

$$\mathrm{g}^2[e_i] = \frac{e_{a_i}^2 + 2e_{a_i}v_i + v_i^2}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}. \quad (11)$$

If we multiply $\mathrm{g}^2[e_i]$ by $||\mathbf{u}_i||^2_{\mathrm{H}}$ from the left and use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, we obtain

$$\begin{aligned}\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\mathrm{g}^2[e_i]\right] &= \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}} e_{a_i}^2}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right] \\ &\quad + \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\sigma_v^2. \quad (12)\end{aligned}$$

Substituting (10) and (12) into (8) we get

$$\begin{aligned}\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}} e_{a_i}^2}{(\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right] + \mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\sigma_v^2 \\ = 2\mathrm{E}\left[\frac{e_{a_i}^2}{\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}}}\right]. \quad (13)\end{aligned}$$

In order to simplify (13), we use the separation principle, namely, that at steady-state, $||\mathbf{u}_i||^2_{\mathrm{H}}$ is independent of $e_{a_i}^2$. Therefore, we obtain

$$\begin{aligned}\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\mathrm{E}[e_{a_i}^2] + \mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\sigma_v^2 \\ = 2\mathrm{E}\left[\frac{1}{\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}}}\right]\mathrm{E}[e_{a_i}^2]. \quad (14)\end{aligned}$$

Now, let us define the following quantities:

$$\mathcal{Z}_1 \triangleq \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right], \quad (15)$$

$$\mathcal{Z}_2 \triangleq \mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\right], \quad (16)$$

then we can write (14) more compactly as

$$\mu\mathcal{Z}_1\mathrm{E}[e_{a_i}^2] + \mu\mathcal{Z}_1\sigma_v^2 = 2\mathcal{Z}_2\mathrm{E}[e_{a_i}^2]. \quad (17)$$

Therefore, the expression for the steady-state excess-mean-square error (EMSE), $\zeta_{\text{NSRLMS}} = \mathrm{E}[e_{a_i}^2]$, of the NSRLMS algorithm is given by

$$\zeta_{\text{NSRLMS}} = \frac{\mu\mathcal{Z}_1\sigma_v^2}{2\mathcal{Z}_2 - \mu\mathcal{Z}_1}. \quad (18)$$

When ϵ is sufficiently small, which is usually the case, then its effect can be ignored. Therefore,

$$\mathcal{Z}_1 = \mathcal{Z}_2 = \mathrm{E}\left[\frac{1}{||\mathbf{u}_i||^2_{\mathrm{H}}}\right]. \quad (19)$$

In this case, expression (18) becomes

$$\zeta_{\text{NSRLMS}} = \frac{\mu\sigma_v^2}{2-\mu}. \quad (20)$$

Finally, the expression for the steady-state MSE, φ_{NSRLMS}, of the NSRLMS algorithm is given below, which is exactly the same as in (7):

$$\varphi_{\text{NSRLMS}} = \frac{\mu\sigma_v^2}{2-\mu} + \sigma_v^2. \quad (21)$$

III. Tracking Analysis

In [2], the expressions for the minimum tracking MSE, $\varphi'_{\min(\text{NSRLMMN})} = \mathrm{E}[e_i^2]$, and the corresponding optimum step-size, $\mu_{\text{opt(NSRLMMN)}}$, of the NSRLMMN algorithm are given by

$$\varphi'_{\min(\text{NSRLMMN})} = \frac{\mu_{\text{opt}} c \mathcal{Z}_1 + \mu_{\text{opt}}^{-1}\mathrm{Tr}(\mathbf{Q})}{2a\mathcal{Z}_2 - \mu_{\text{opt}} b \mathcal{Z}_1} + \sigma_v^2, \tag{22}$$

$$\mu_{\text{opt(NSRLMMN)}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{c\mathcal{Z}_1}\left[1 + \frac{b^2\mathcal{Z}_1\mathrm{Tr}(\mathbf{Q})}{4a^2c\mathcal{Z}_2^2}\right]} - \frac{b\mathrm{Tr}(\mathbf{Q})}{2ac\mathcal{Z}_2}, \tag{23}$$

where $a = \delta + 3\bar{\delta}\sigma_v^2$, $b = \delta^2 + 15\bar{\delta}^2\xi_v^4 + 12\delta\bar{\delta}\sigma_v^2$, $c = \delta^2\sigma_v^2 + \bar{\delta}^2\xi_v^6 + 2\delta\bar{\delta}\xi_v^4$, and $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^T]$ is the covariance matrix of the noise sequence $\mathbf{q}_i$.

The expressions for the minimum tracking MSE and the corresponding optimum step-size of the NSRLMF [4] and NSRLMS algorithms can be easily obtained by substituting δ is equal to 0 and 1 in both (22) and (23), respectively, as shown below:

$$\varphi'_{\min(\text{NSRLMF})} = \frac{\mu_{\text{opt}}\mathcal{Z}_1\xi_v^6 + \mu_{\text{opt}}^{-1}\mathrm{Tr}(\mathbf{Q})}{6\sigma_v^2\mathcal{Z}_2 - 15\mu_{\text{opt}}\mathcal{Z}_1\xi_v^4} + \sigma_v^2, \tag{24}$$

$$\mu_{\text{opt(NSRLMF)}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\xi_v^6}\left[\frac{25(\xi_v^4)^2\mathrm{Tr}(\mathbf{Q})}{4\sigma_v^4(\mathcal{Z}_2)^2\xi_v^6} + \frac{1}{\mathcal{Z}_1}\right]} - \frac{5\xi_v^4\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2\mathcal{Z}_2\xi_v^6}, \tag{25}$$

$$\varphi'_{\min(\text{NSRLMS})} = \frac{\mu_{\text{opt}}\mathcal{Z}_1\sigma_v^2 + \mu_{\text{opt}}^{-1}\mathrm{Tr}(\mathbf{Q})}{2\mathcal{Z}_2 - \mu_{\text{opt}}\mathcal{Z}_1} + \sigma_v^2, \tag{26}$$

$$\mu_{\text{opt(NSRLMS)}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2\mathcal{Z}_1}\left[1 + \frac{\mathcal{Z}_1\mathrm{Tr}(\mathbf{Q})}{4\sigma_v^2\mathcal{Z}_2^2}\right]} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2\mathcal{Z}_2}. \tag{27}$$

Note that the expressions for the minimum tracking MSE and the corresponding optimum step-size of the NSRLMF algorithm in (24) and (25), respectively, are exactly the same as the one derived by us in [4].

To carry out the tracking analysis of the NSRLMS algorithm we shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the nonstationary data model [2], [4], [9]– [11]:

- A.5 There exists a vector $\mathbf{w}_i^o$ such that $d_i = \mathbf{u}_i\mathbf{w}_i^o + v_i$.
- A.6 The weight vector varies according to the random-walk model $\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i$, and the noise sequence $\mathbf{q}_i$ is i.i.d. with variance $\sigma_q^2 = \mathrm{E}[\mathbf{q}_i^2]$. Moreover, $\mathbf{q}_i$ is independent of $\{\mathbf{u}_j, v_j\}$ for all i, j.
- A.7 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

For the adaptive filter of the form in (4), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [2], [4], [9]– [11]:

$$\mu\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 \mathrm{g}^2[e_i]\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \to \infty. \tag{28}$$

We can obtain the tracking results by inspecting the steady-state results in Section II as there are only minor differences. Therefore, by substituting (10) and (12) into (28) the expression for the tracking EMSE, $\zeta'_{\text{NSRLMS}} = \mathrm{E}[e_{a_i}^2]$, of the NSRLMS algorithm can be shown to be

$$\zeta'_{\text{NSRLMS}} = \frac{\mu\mathcal{Z}_1\sigma_v^2 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\mathcal{Z}_2 - \mu\mathcal{Z}_1}. \tag{29}$$

Consequently, the optimum step-size, $\mu_{\text{opt(NSRLMS)}}$, of the NSRLMS algorithm can be obtained by minimizing (29) with respect to μ and is given by

$$\mu_{\text{opt(NSRLMS)}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2\mathcal{Z}_1}\left[1 + \frac{\mathcal{Z}_1\mathrm{Tr}(\mathbf{Q})}{4\sigma_v^2\mathcal{Z}_2^2}\right]} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2\mathcal{Z}_2}. \tag{30}$$

Finally, the corresponding minimum tracking MSE, $\varphi'_{\min(\text{NSRLMS})}$, of the NSRLMS algorithm is given by

$$\varphi'_{\min(\text{NSRLMS})} = \frac{\mu_{\text{opt}}\mathcal{Z}_1\sigma_v^2 + \mu_{\text{opt}}^{-1}\mathrm{Tr}(\mathbf{Q})}{2\mathcal{Z}_2 - \mu_{\text{opt}}\mathcal{Z}_1} + \sigma_v^2. \tag{31}$$

Note that the expressions for the optimum step-size and the corresponding minimum tracking MSE of the NSRLMS algorithm in (30) and (31), respectively, are exactly the same as the one in (27) and (26).

IV. Simulation Results

The parameter settings in this study are as follows. In all the figures, ϵ is fixed at 10^{-6} and M is fixed at 5. σ_q^2 is fixed at 10^{-6} in Figures 3–4.

The steady-state performance of the NSRLMS algorithm using white and correlated Gaussian regressors is shown in Figures 1–2, respectively. In Figures 2 and 4, the correlated data is obtained by passing a unit-variance i.i.d. Gaussian data through a first-order auto-regressive model with transfer function $\frac{\sqrt{1-a^2}}{(1-az^{-1})}$ and $a = 0.8$. As can be seen from Figures 1–2, the simulation result and the theoretical result in (21) are in close agreement for correlated Gaussian regressors than white Gaussian regressors.

Figures 3–4 show the tracking performance of the NSRLMS algorithm using white and correlated Gaussian regressors, respectively. A zoom into the region around $\mu = 0.04$ in Figure 3 shows that the tracking

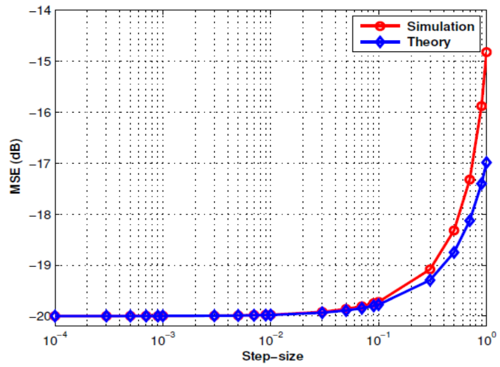

Fig. 1. Theoretical and simulated steady-state MSE of the NSRLMS algorithm using white Gaussian regressors in an AWGN environment at SNR = 20 dB.

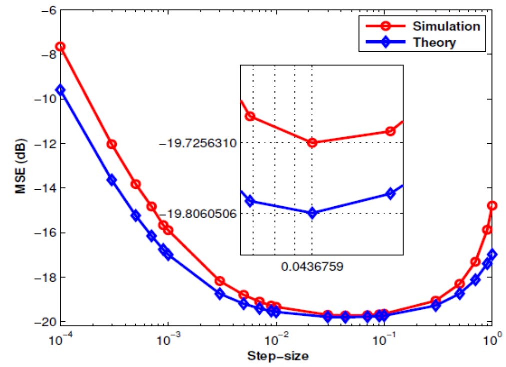

Fig. 3. Theoretical and simulated tracking MSE of the NSRLMS algorithm using white Gaussian regressors in an AWGN environment at SNR = 20 dB.

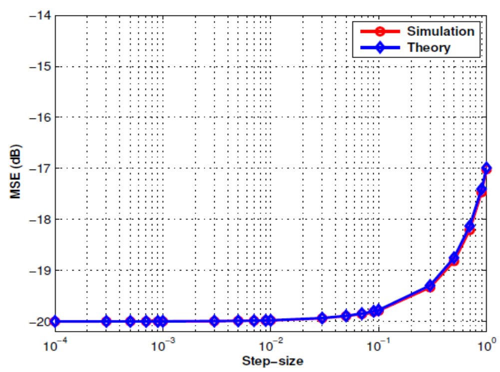

Fig. 2. Theoretical and simulated steady-state MSE of the NSRLMS algorithm using correlated Gaussian regressors in an AWGN environment at SNR = 20 dB.

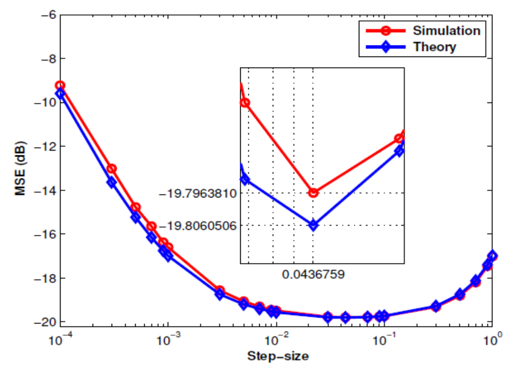

Fig. 4. Theoretical and simulated tracking MSE of the NSRLMS algorithm using correlated Gaussian regressors in an AWGN environment at SNR = 20 dB.

MSE possesses a minimum value of -19.7256310 dB at $\mu = 0.04$, which are in good agreement with the corresponding theoretical values of $\varphi'_{\min(\text{NSRLMS})} = -19.8060506$ dB at $\mu_{\text{opt(NSRLMS)}} = 0.0436759$ obtained from expressions (31) and (30), respectively. Similarly, a zoom into the region around $\mu = 0.04$ in Figure 4 shows that the tracking MSE possesses a minimum value of -19.7963810 dB at $\mu = 0.04$, which are in good agreement with the corresponding theoretical values of $\varphi'_{\min(\text{NSRLMS})} = -19.8060506$ dB at $\mu_{\text{opt(NSRLMS)}} = 0.0436759$ obtained from expressions (31) and (30), respectively.

Finally, Figure 5 shows the convergence performance of the NSRLMMN algorithm for different values of δ. As can be seen from this figure, the NSRLMMN algorithm reduces to NSRLMF and NSRLMS algorithms when δ takes the value 0 and 1, respectively.

V. Conclusions

In this paper, we have shown that the expressions for the steady-state MSE, optimum step-size, and the corresponding minimum tracking MSE of the NSRLMF algorithm, which are derived in [3]– [4] are exactly the same as those obtained by substituting δ is equal to 0 in the corresponding expressions of the NSRLMMN algorithm, which are derived in [1]–[2]. Furthermore, in this paper we have derived the expressions for the steady-state MSE, optimum step-size, and the corresponding minimum tracking MSE of the NSRLMS algorithm, which are found to be exactly the same as those obtained by substituting δ is equal to 1 in the corresponding expressions of the NSRLMMN algorithm, which are derived in [1]–[2]. Finally, we observe a close agreement between

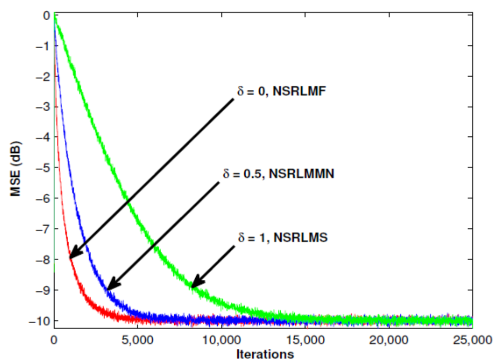

Fig. 5. Comparison of the MSE learning curves of the NSRLMMN algorithm for different values of δ in an AWGN environment at SNR = 10 dB.

the simulation and analytical results of the NSRLMS algorithm for correlated Gaussian regressors than white Gaussian regressors.

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[P17]

A STEADY-STATE ANALYSIS OF THE $\varepsilon-$NORMALIZED SIGN-ERROR LEAST MEAN SQUARE (NSLMS) ADAPTIVE ALGORITHM

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ABSTRACT

In this work, expressions are derived for the steady-state excess-mean-square error (EMSE) of the $\varepsilon-$normalized sign-error least mean square (NSLMS) adaptive algorithm for both cases of real- and complex-valued data. Moreover, a comparison between the computational load of the $\varepsilon-$NSLMS algorithm and the $\varepsilon-$normalized least mean square (NLMS) algorithm is also presented. Finally, simulation results to substantiate the theoretical findings are presented.

1. INTRODUCTION

The sign based variants of the least mean square (LMS) algorithm [1] were introduced with the objective to reduce its computational and implementation costs [2]–[3]. The sign-error algorithm or simply the sign algorithm (SA) as it is more commonly known is one such variant of the LMS algorithm, which is based on clipping of the estimation error [4]. However, these sign based algorithms result in a performance loss in terms of convergence behavior when compared with the LMS algorithm [5]–[6].

In [7], it is shown that the normalized least mean square (NLMS) algorithm exhibits faster convergence than the LMS algorithm. In [8], the convergence analysis of the normalized sign-error least mean square (NSLMS) algorithm was provided and the algorithm was tested in an adaptive noise cancellation scenario. In this paper, expressions are evaluated for the steady-state excess-mean-square error (EMSE) of the $\varepsilon-$NSLMS algorithm in a stationary environment. The framework used in this paper relies on energy conservation arguments [9]. From the simulation results it is shown that the theoretical and simulated results are in good agreement.

The paper is organized as follows: following the Introduction is Section 2 where the $\varepsilon-$NSLMS algorithm is described, while Section 3 deals with the mean-square analysis. Section 4 details the computational load of the $\varepsilon-$NSLMS algorithm. The simulation results are reported in Section 5. Finally, some conclusions are reported in Section 6.

2. THE $\varepsilon-$NSLMS ALGORITHM

Consider a zero-mean random variable d with realizations $\{d_0, d_1, \ldots\}$, and a zero-mean random row vector $\mathbf{u}$ with realizations $\{\mathbf{u}_0, \mathbf{u}_1, \ldots\}$. The optimal weight vector $\mathbf{w}^o$ that solves

$$\min_{\mathbf{w}_i} ||\mathbf{w}_i - \mathbf{w}_{i-1}||^2, \tag{1}$$

subject to

$$d_i - \mathbf{u}_i \mathbf{w}_i = 0, \tag{2}$$

can be approximated iteratively via the recursion (the $\varepsilon-$NSLMS algorithm)

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\varepsilon + ||\mathbf{u}_i||^2} \mathbf{u}_i^{\mathrm{T}} \mathrm{sign}[e_i], \quad i \geq 0, \tag{3}$$

where $\mathbf{w}_i$ (column vector) is the updated weight vector at time i, μ is the step-size, ε is a small positive constant to avoid division by zero when the regressor is zero, and e_i denotes the estimation error given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}. \tag{4}$$

For complex-valued data, the update recursion in (3) becomes

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\varepsilon + ||\mathbf{u}_i||^2} \mathbf{u}_i^{*} \mathrm{csgn}[e_i], \quad i \geq 0. \tag{5}$$

3. MEAN-SQUARE ANALYSIS OF THE $\varepsilon-$NSLMS ALGORITHM

In this section, the mean-square analysis of the $\varepsilon-$NSLMS algorithm for both cases of real- and complex-valued data is carried out. In the process of this evaluation, we distinguished between real- and complex-valued data as the definition of the sign function is different in both cases.

We shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the stationary data model [9]:

A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that $d_i = \mathbf{u}_i \mathbf{w}^o + v_i$.

A.2 The noise sequence v_i is independent and identically distributed (i.i.d.) with variance $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ and is independent of $\mathbf{u}_j$ for all i, j.

A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.

A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^* \mathbf{u}_i] > \mathbf{0}$.

3.1 Real-Valued Data

For the adaptive filter of the form in (3), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [9]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||^2 \mathrm{g}^2[e_i]\right] = 2\mathrm{E}\left[e_{a_i} \mathrm{g}[e_i]\right], \text{ as } i \to \infty, \tag{6}$$

where

$$e_i = e_{a_i} + v_i, \tag{7}$$

with $\mathrm{g}[.]$ denoting some function of e_i, and $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error. Then $\mathrm{g}[e_i]$ for the

$\varepsilon-$NSLMS algorithm becomes

$$g[e_i] = \frac{\text{sign}[e_{a_i} + v_i]}{\varepsilon + ||\mathbf{u}_i||^2}. \tag{8}$$

Substituting (8) into (6) we get

$$\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2(\text{sign}[e_{a_i}+v_i])^2}{(\varepsilon+||\mathbf{u}_i||^2)^2}\right] = 2\mathrm{E}\left[\frac{e_{a_i}\text{sign}[e_{a_i}+v_i]}{\varepsilon+||\mathbf{u}_i||^2}\right]. \tag{9}$$

Using the fact that $(\text{sign}[x])^2 = 1$ almost everywhere on the real line, we get

$$\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\varepsilon+||\mathbf{u}_i||^2)^2}\right] = 2\mathrm{E}\left[\frac{e_{a_i}\text{sign}[e_{a_i}+v_i]}{\varepsilon+||\mathbf{u}_i||^2}\right]. \tag{10}$$

In order to simplify (10), we resort to the separation principle, namely, that at steady-state, $||\mathbf{u}_i||^2$ is independent of e_{a_i}. We then obtain

$$\begin{aligned}\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\varepsilon+||\mathbf{u}_i||^2)^2}\right] &= 2\mathrm{E}\left[\frac{1}{\varepsilon+||\mathbf{u}_i||^2}\right] \\ &\times \mathrm{E}\left[e_{a_i}\text{sign}[e_{a_i}+v_i]\right]. \end{aligned} \tag{11}$$

If we define the following quantities:

$$\alpha_u \triangleq \mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\varepsilon+||\mathbf{u}_i||^2)^2}\right], \tag{12}$$

$$\eta_u \triangleq \mathrm{E}\left[\frac{1}{\varepsilon+||\mathbf{u}_i||^2}\right], \tag{13}$$

then the equality in (11) can be written more compactly as

$$\mu\alpha_u = 2\eta_u \mathrm{E}\left[e_{a_i}\text{sign}[e_{a_i}+v_i]\right]. \tag{14}$$

In order to evaluate the expectation on the right-hand side of (14) let us rely on the following assumption [9]:

A.5 The estimation error e_i and the noise sequence v_i are jointly Gaussian.

From Price's theorem [10] for real-valued data we have

$$\mathrm{E}[x\ \text{sign}(y)] = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma_y}\mathrm{E}[xy], \tag{15}$$

then

$$\mathrm{E}\left[e_{a_i}\text{sign}[e_{a_i}+v_i]\right] = \sqrt{\frac{2}{\pi}}\left(\frac{\mathrm{E}[e_{a_i}^2]}{\sqrt{\mathrm{E}[e_{a_i}^2]+\sigma_v^2}}\right). \tag{16}$$

Substituting (16) into (14) we get

$$\mu\alpha_u = 2\eta_u\sqrt{\frac{2}{\pi}}\left(\frac{\mathrm{E}[e_{a_i}^2]}{\sqrt{\mathrm{E}[e_{a_i}^2]+\sigma_v^2}}\right). \tag{17}$$

When the regularization parameter ε is sufficiently small, which is usually the case, then its effect can be ignored and the definitions of α_u and η_u coincide, and in this case (17) reduces to

$$\mu = 2\sqrt{\frac{2}{\pi}}\left(\frac{\mathrm{E}[e_{a_i}^2]}{\sqrt{\mathrm{E}[e_{a_i}^2]+\sigma_v^2}}\right). \tag{18}$$

Therefore, the expression for the steady-state EMSE $\zeta = \mathrm{E}[e_{a_i}^2]$ of the $\varepsilon-$NSLMS algorithm is given by

$$\zeta = \frac{\mu}{4}\sqrt{\frac{\pi}{2}}\left[\frac{\mu}{2}\sqrt{\frac{\pi}{2}}+\sqrt{\frac{\mu^2\pi}{8}+4\sigma_v^2}\right]. \tag{19}$$

3.2 Complex-Valued Data

For the adaptive filter of the form in (5), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [9]:

$$\mu\mathrm{E}\left[||\mathbf{u}_i||^2|g[e_i]|^2\right] = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^* g[e_i]\right]\right], \text{ as } i\to\infty. \tag{20}$$

In this case, $g[e_i]$ for the $\varepsilon-$NSLMS algorithm becomes

$$g[e_i] = \frac{\text{csgn}[e_{a_i}+v_i]}{\varepsilon+||\mathbf{u}_i||^2}. \tag{21}$$

Substituting (21) into (20), and using the fact that $|\text{csgn}[x]|^2 = 2$ almost everywhere in the complex plane, we get

$$\mu\mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\varepsilon+||\mathbf{u}_i||^2)^2}\right] = \mathrm{Re}\left[\mathrm{E}\left[\frac{e_{a_i}^*\text{csgn}[e_{a_i}+v_i]}{\varepsilon+||\mathbf{u}_i||^2}\right]\right]. \tag{22}$$

Therefore, proceeding on the same lines as we did in the previous section following (10), we obtain

$$\mu\alpha_u = \eta_u\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^*\text{csgn}[e_{a_i}+v_i]\right]\right]. \tag{23}$$

In order to evaluate the expectation on the right-hand side of (23) let us rely on the following assumptions [9]:

A.6 The real parts of e_i and v_i are jointly Gaussian.

A.7 The imaginary parts of e_i and v_i are jointly Gaussian.

A.8 The real and imaginary parts of e_i have identical variances.

A.9 The real parts of $\{e_i, v_i\}$ are independent of their imaginary parts.

From Price's theorem [10] for complex-valued data we have

$$\mathrm{E}\left[\mathrm{Re}[x^*\text{csgn}(y)]\right] = \frac{2}{\sqrt{\pi}\sigma_y}\mathrm{E}\left[\mathrm{Re}[x^*y]\right], \tag{24}$$

then

$$\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^*\text{csgn}[e_{a_i}+v_i]\right]\right] = \frac{2}{\sqrt{\pi}}\left(\frac{\mathrm{E}[|e_{a_i}|^2]}{\sqrt{\mathrm{E}[|e_{a_i}|^2]+\sigma_v^2}}\right). \tag{25}$$

Substituting (25) into (23), and using the fact that the regularization parameter ε is sufficiently small, we get

$$\mu = \frac{2}{\sqrt{\pi}}\left(\frac{\mathrm{E}[|e_{a_i}|^2]}{\sqrt{\mathrm{E}[|e_{a_i}|^2]+\sigma_v^2}}\right). \tag{26}$$

Also, here, in this case we get

$$\zeta = \frac{\mu\sqrt{\pi}}{4}\left[\frac{\mu\sqrt{\pi}}{2}+\sqrt{\frac{\mu^2\pi}{4}+4\sigma_v^2}\right]. \tag{27}$$

It is interesting to note that the expressions ((19) and (27)) for the steady-state EMSE of the $\varepsilon-$NSLMS algorithm, respectively, for real- and complex-valued data are identical except for a scaling factor. Also, the EMSE is found to be independent of the regression data. Moreover, in both cases, the expression for the MSE of the $\varepsilon-$NSLMS algorithm is obtained by

$$\mathrm{E}\left[|e_i|^2\right] = \zeta + \sigma_v^2. \tag{28}$$

4. COMPUTATIONAL LOAD

Finally, the computational load of the $\varepsilon-$NLMS and the $\varepsilon-$NSLMS algorithms is discussed in this section. Tables 1 and 2 detail the estimated computational load per iteration of the $\varepsilon-$NLMS and the $\varepsilon-$NSLMS algorithms, respectively, for real- and complex-valued data in terms of the number of real additions $(+)$, real multiplications $(\times)$, real divisions $(/)$, and sign evaluations.

Table 1: Computational load per iteration of the $\varepsilon-$NLMS and the $\varepsilon-$NSLMS algorithms for real-valued data.

Algorithm	$+$	$\times$	$/$	**sign**
$\varepsilon-$NLMS	$3M$	$3M+1$	1	
$\varepsilon-$NSLMS	$3M$	$3M$	1	1

Table 2: Computational load per iteration of the $\varepsilon-$NLMS and the $\varepsilon-$NSLMS algorithms for complex-valued data.

Algorithm	$+$	$\times$	$/$	**sign**
$\varepsilon-$NLMS	$10M$	$10M+2$	1	
$\varepsilon-$NSLMS	$8M$	$8M$	2	2

As can be seen from Table 1, the computational load per iteration of the $\varepsilon-$NSLMS algorithm is similar to that of the $\varepsilon-$NLMS algorithm for real-valued data. However, significant reduction in computational load per iteration is achieved for the $\varepsilon-$NSLMS algorithm over the $\varepsilon-$NLMS algorithm for complex-valued data as seen from Table 2.

5. SIMULATION RESULTS

In order to validate the theoretical findings extensive simulations are carried out for different scenarios. While Figures 1-3 are for the case of real-valued data and Figures 4-6 are for the case of complex-valued data. In all of these figures the MSE of a 5-tap $\varepsilon-$NSLMS filter is plotted as a function of the step-size μ for a signal to noise ratio (SNR) of 30 dB and the value of ε is set to 10^{-6}.

In the case of Figures 1 and 4 the regressors, with shift structure, are generated by feeding a unit-variance white process into a tapped delay line. However, in Figures 2 and 5 the regressors, with shift structure, are generated by passing correlated data into a tapped delay line. Here, the correlated data is obtained by passing a unit-variance i.i.d. Gaussian data through a first-order auto-regressive model with transfer function $\frac{\sqrt{1-a^2}}{(1-az^{-1})}$ and $a = 0.8$. To further test the validity of the results, Gaussian regressors with an eigenvalue spread of five without a shift structure are used, this is depicted in Figures 3 and 6. As can be seen from these figures, the simulation results match very well the theoretical results ((19) and (27)).

6. CONCLUSIONS

Expressions derived for the steady-state EMSE of the $\varepsilon-$NSLMS algorithm are found to be identical for real- and complex-valued data except for a scaling factor. Moreover, it is shown that the $\varepsilon-$NSLMS algorithm is computationally more simple than the $\varepsilon-$NLMS algorithm for complex-valued data. Finally, simulations performed are found to corroborate with the theory developed.

7. ACKNOWLEDGMENT

The authors acknowledge the support provided by King Fahd University of Petroleum and Minerals to carry out this work.

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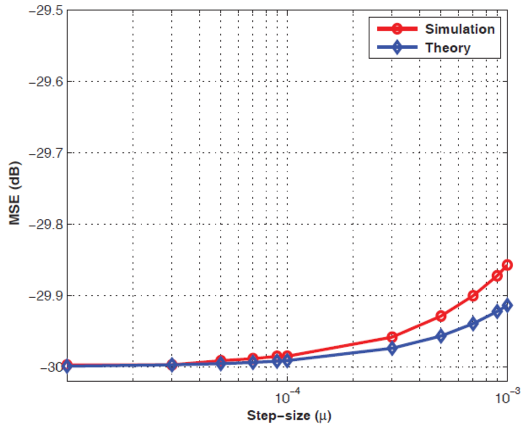

Figure 1: Theoretical and simulated MSE of the $\varepsilon-$NSLMS algorithm using white Gaussian regressors.

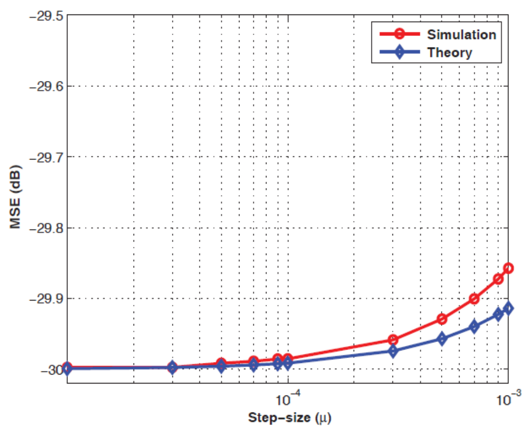

Figure 2: Theoretical and simulated MSE of the $\varepsilon-$NSLMS algorithm using correlated Gaussian regressors.

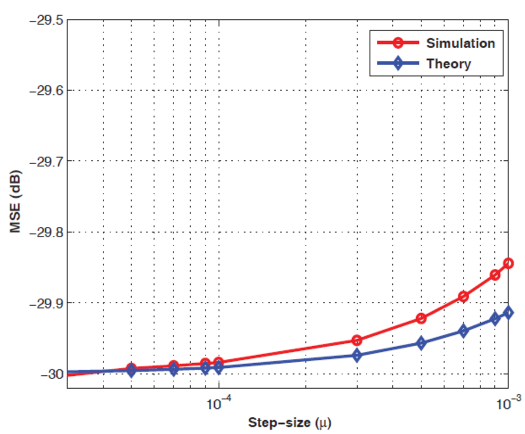

Figure 3: Theoretical and simulated MSE of the $\varepsilon-$NSLMS algorithm using Gaussian regressors with an eigenvalue spread=5.

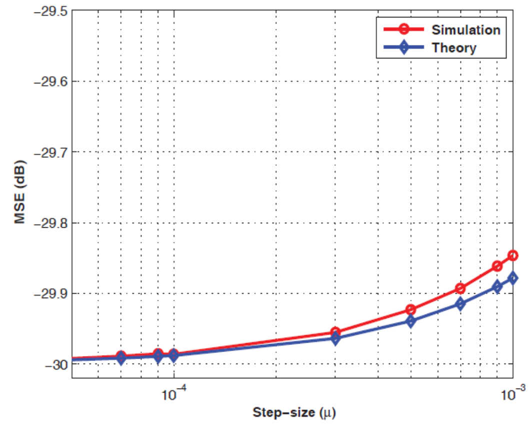

Figure 4: Theoretical and simulated MSE of the complex $\varepsilon-$NSLMS algorithm using white Gaussian regressors.

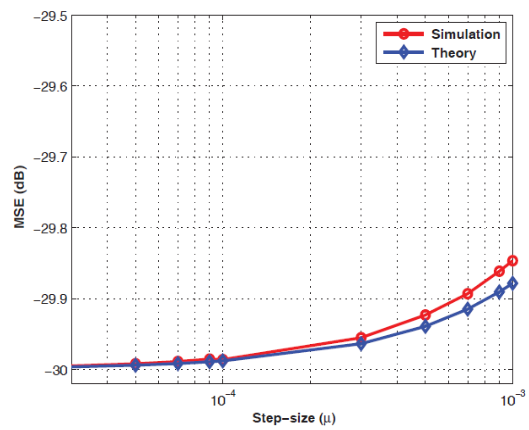

Figure 5: Theoretical and simulated MSE of the complex $\varepsilon-$NSLMS algorithm using correlated Gaussian regressors.

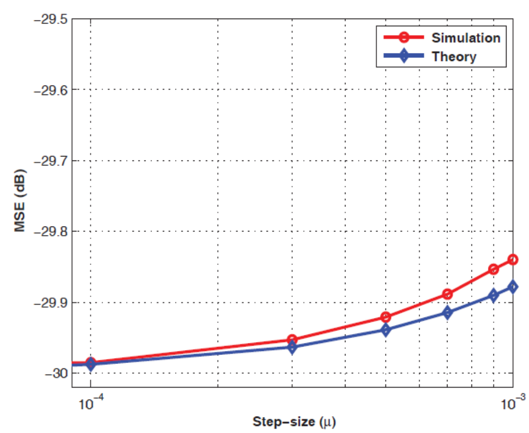

Figure 6: Theoretical and simulated MSE of the complex $\varepsilon-$NSLMS algorithm using Gaussian regressors with an eigenvalue spread=5.

[P18]

Tracking MSE Performance Analysis of the $\epsilon-$NSLMS Algorithm

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***Abstract*—In this paper, the tracking behavior of the $\epsilon-$normalized sign-error least mean square (NSLMS) algorithm is analyzed in the presence of white and correlated Gaussian regressors. Moreover, generic analytical expressions are derived for the optimal step-size and the corresponding optimal mean-square error (MSE) of the $\epsilon-$NSLMS algorithm for both the real- and complex-valued data cases. Additionally, a comparison between the convergence behavior of the $\epsilon-$NSLMS algorithm and the $\epsilon-$normalized least mean square (NLMS) algorithm is also discussed. Finally, simulation results to corroborate our theoretical findings are presented.**

***Keywords*—LMS, NLMS, SLMS, NSLMS, Tracking.**

I. Introduction

The $\epsilon-$normalized sign-error least mean square algorithm or simply the $\epsilon-$normalized sign least mean square (NSLMS) algorithm belongs to the families of normalized least mean square (NLMS) algorithms and sign algorithms (SA) and is based on clipping of the estimation error signal [1]–[2]. The $\epsilon-$NSLMS algorithm is computationally more simple than the $\epsilon-$NLMS algorithm due to the clipping of the error signal [3]. In [4], it is shown that the $\epsilon-$NSLMS algorithm performs better than the least mean square (LMS) algorithm to eliminate noise from the electrocardiogram (ECG) signal.

In [5], the convergence analysis of the NSLMS algorithm was performed for an adaptive noise cancellation system. In our previous paper [3], the steady-state analysis of the $\epsilon-$NSLMS was performed and we showed that the mean-square error (MSE) expressions of the $\epsilon-$NSLMS algorithm for real- and complex-valued data are identical except for a scaling factor.

In the present paper general analytical expressions are derived for the tracking analysis of the $\epsilon-$NSLMS algorithm based on the energy conservation approach [3], [6]. Moreover, the analytical expressions derived are found to be in good agreement with the simulation results.

The rest of the paper is organized as follows. In Section 2, the $\epsilon-$NSLMS algorithm is described. In Section 3, the tracking analysis of the $\epsilon-$NSLMS algorithm is derived. Simulation results are presented in Section 4, followed by conclusions in Section 5.

II. The $\epsilon-$NSLMS Algorithm

The weight update recursion of the $\epsilon-$NSLMS algorithm for real-valued data is given by the following expression:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2}\, \mathbf{u}_i^{\mathrm{T}} \mathrm{sign}[e_i], \quad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ is the updated weight vector, μ is the step-size, ϵ is a small positive constant to avoid division by zero when the regressor $\mathbf{u}_i$ is zero, d_i is the desired value, and e_i is the estimation error signal given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}. \tag{2}$$

For complex-valued data, the update recursion in (1) becomes

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2}\, \mathbf{u}_i^{*} \mathrm{csgn}[e_i], \quad i \geq 0. \tag{3}$$

III. Tracking Analysis

The main aim of the tracking analysis of an adaptive filter is to quantify its ability to track the time variations in the channel. In this section, the tracking analysis of the $\epsilon-$NSLMS algorithm is carried out in a straightforward manner using its steady-state analysis presented in [3] as there are only minor differences. While performing this analysis, we differentiate between real- and complex-valued data as the sign function is defined differently for each case [7].

Here, let us assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the nonstationary data model [6], [8]–[9]:

A.1 There exists an optimal weight vector $\mathbf{w}_i^o$ such that

$$d_i = \mathbf{u}_i \mathbf{w}_i^o + v_i, \tag{4}$$

where v_i is the additive noise with variance σ_v^2.

A.2 The weight vector varies according to the random-walk model

$$\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i, \tag{5}$$

and the Gaussian noise sequence $\mathbf{q}_i$ is independent and identically distributed (i.i.d.) with variance σ_q^2 and covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.

A.3 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

A. Real-Valued Data

The following variance relation holds for the weight update recursion of the ϵ−NSLMS algorithm given in (1) when the data is real-valued [6]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||^2 \mathrm{g}^2[e_i]\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \to \infty, \tag{6}$$

where

$$e_i = e_{a_i} + v_i, \tag{7}$$

with $\mathrm{g}[e_i]$ denoting some function of e_i, $e_{a_i} = \mathbf{u}_i(\mathbf{w}_i^o - \mathbf{w}_{i-1})$ is the a priori estimation error, and when the data is real-valued $\mathrm{g}[e_i]$ for the ϵ−NSLMS algorithm is given by

$$\mathrm{g}[e_i] = \frac{\mathrm{sign}[e_{a_i} + v_i]}{\epsilon + ||\mathbf{u}_i||^2}. \tag{8}$$

Substituting (8) into (6) we get

$$\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2(\mathrm{sign}[e_{a_i} + v_i])^2}{(\epsilon + ||\mathbf{u}_i||^2)^2}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left[\frac{e_{a_i}\mathrm{sign}[e_{a_i} + v_i]}{\epsilon + ||\mathbf{u}_i||^2}\right]. \tag{9}$$

Since $(\mathrm{sign}[x])^2 = 1$, we get

$$\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\epsilon + ||\mathbf{u}_i||^2)^2}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left[\frac{e_{a_i}\mathrm{sign}[e_{a_i} + v_i]}{\epsilon + ||\mathbf{u}_i||^2}\right]. \tag{10}$$

Using the separation principle [3], we get

$$\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\epsilon + ||\mathbf{u}_i||^2)^2}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||^2}\right]\mathrm{E}\left[e_{a_i}\mathrm{sign}[e_{a_i} + v_i]\right]. \tag{11}$$

The above equation can be rewritten as

$$\mu \mathcal{Z}_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathcal{Z}_2 \mathrm{E}\left[e_{a_i}\mathrm{sign}[e_{a_i} + v_i]\right], \tag{12}$$

where

$$\mathcal{Z}_1 \triangleq \mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\epsilon + ||\mathbf{u}_i||^2)^2}\right], \tag{13}$$

and

$$\mathcal{Z}_2 \triangleq \mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||^2}\right]. \tag{14}$$

From [3], we have

$$\mathrm{E}\left[e_{a_i}\mathrm{sign}[e_{a_i} + v_i]\right] = \sqrt{\frac{2}{\pi}}\left(\frac{\mathrm{E}[e_{a_i}^2]}{\sqrt{\mathrm{E}[e_{a_i}^2] + \sigma_v^2}}\right). \tag{15}$$

Substituting (15) into (12) we get

$$\mu \mathcal{Z}_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathcal{Z}_2\sqrt{\frac{2}{\pi}}\left(\frac{\mathrm{E}[e_{a_i}^2]}{\sqrt{\mathrm{E}[e_{a_i}^2] + \sigma_v^2}}\right). \tag{16}$$

After some simplification, the expression for the tracking excess-mean-square error (EMSE), $\zeta = \mathrm{E}[e_{a_i}^2]$, of the ϵ−NSLMS algorithm is given by

$$\zeta = \frac{\gamma\sqrt{\pi}}{4\mathcal{Z}_2^2}\left[\frac{\gamma\sqrt{\pi}}{4} + \sqrt{\frac{\gamma^2\pi}{16} + 2\sigma_v^2\mathcal{Z}_2^2}\right], \tag{17}$$

where

$$\gamma = \mu\mathcal{Z}_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q}). \tag{18}$$

Consequently, the optimum step-size of the ϵ−NSLMS algorithm can be obtained by minimizing (18) with respect to μ and is given by

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\mathcal{Z}_1}}. \tag{19}$$

Finally, the corresponding minimum value of the tracking MSE, $\varphi = \mathrm{E}\left[e_i^2\right]$, of the ϵ−NSLMS algorithm for real-valued data can be shown to be

$$\varphi_{\min} = \frac{\gamma_{\mathrm{opt}}\sqrt{\pi}}{4\mathcal{Z}_2^2}\left[\frac{\gamma_{\mathrm{opt}}\sqrt{\pi}}{4} + \sqrt{\frac{\gamma_{\mathrm{opt}}^2\pi}{16} + 2\sigma_v^2\mathcal{Z}_2^2}\right] + \sigma_v^2, \tag{20}$$

where

$$\gamma_{\mathrm{opt}} = \mu_{\mathrm{opt}}\mathcal{Z}_1 + \mu_{\mathrm{opt}}^{-1}\mathrm{Tr}(\mathbf{Q}). \tag{21}$$

B. Complex-Valued Data

In this case, the following variance relation holds for the weight update recursion of the ϵ−NSLMS algorithm given in (3) when the data is complex-valued [6]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||^2 |\mathrm{g}[e_i]|^2\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^*\mathrm{g}[e_i]\right]\right], \quad \text{as } i \to \infty. \tag{22}$$

For complex-valued data, $\mathrm{g}[e_i]$ for the ϵ−NSLMS algorithm is given by

$$\mathrm{g}[e_i] = \frac{\mathrm{csgn}[e_{a_i} + v_i]}{\epsilon + ||\mathbf{u}_i||^2}. \tag{23}$$

Substituting (23) into (22), and since $|\mathrm{csgn}[x]|^2 = 2$, we get

$$2\mu \mathrm{E}\left[\frac{||\mathbf{u}_i||^2}{(\epsilon + ||\mathbf{u}_i||^2)^2}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{Re}\left[\mathrm{E}\left[\frac{e_{a_i}^*\mathrm{csgn}[e_{a_i} + v_i]}{\epsilon + ||\mathbf{u}_i||^2}\right]\right]. \tag{24}$$

Again relying on the separation principle [3], it can be shown that

$$2\mu\mathcal{Z}_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathcal{Z}_2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^*\mathrm{csgn}[e_{a_i} + v_i]\right]\right]. \tag{25}$$

In [3], we have evaluated the following expression for the expectation on the right-hand side of (25)

$$\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^*\mathrm{csgn}[e_{a_i} + v_i]\right]\right] = \frac{2}{\sqrt{\pi}}\left(\frac{\mathrm{E}[|e_{a_i}|^2]}{\sqrt{\mathrm{E}[|e_{a_i}|^2] + \sigma_v^2}}\right). \tag{26}$$

Substituting (26) into (25) we get

$$2\mu\mathcal{Z}_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = \frac{4\mathcal{Z}_2}{\sqrt{\pi}}\left(\frac{\mathrm{E}[|e_{a_i}|^2]}{\sqrt{\mathrm{E}[|e_{a_i}|^2] + \sigma_v^2}}\right). \tag{27}$$

Also, here, in this case we get

$$\zeta = \frac{\gamma\sqrt{\pi}}{8\mathcal{Z}_2^2}\left[\frac{\gamma\sqrt{\pi}}{4} + \sqrt{\frac{\gamma^2\pi}{16} + 4\sigma_v^2\mathcal{Z}_2^2}\right], \tag{28}$$

where

$$\gamma = 2\mu\mathcal{Z}_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q}). \tag{29}$$

Similarly, the optimum step-size of the $\epsilon-$NSLMS algorithm can be obtained by minimizing (29) with respect to μ and is given by

$$\mu_{\text{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{2\mathcal{Z}_1}}. \tag{30}$$

Finally, the corresponding minimum value of the tracking MSE of the $\epsilon-$NSLMS algorithm for complex-valued data can be shown to be

$$\varphi_{\min} = \frac{\gamma_{\text{opt}}\sqrt{\pi}}{8\mathcal{Z}_2^2}\left[\frac{\gamma_{\text{opt}}\sqrt{\pi}}{4} + \sqrt{\frac{\gamma_{\text{opt}}^2\pi}{16} + 4\sigma_v^2\mathcal{Z}_2^2}\right] + \sigma_v^2, \tag{31}$$

where

$$\gamma_{\text{opt}} = 2\mu_{\text{opt}}\mathcal{Z}_1 + \mu_{\text{opt}}^{-1}\mathrm{Tr}(\mathbf{Q}). \tag{32}$$

C. Generalization

The expressions obtained for the optimum step-size of the $\epsilon-$NSLMS algorithm in the previous two sections can be generalized as follows:

$$\mu_{\text{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\alpha\mathcal{Z}_1}}, \tag{33}$$

where the scaling factor $\alpha = 1$ for real-valued data case and $\alpha = 2$ for complex-valued data case. Similarly, the corresponding minimum value of the tracking MSE of the $\epsilon-$NSLMS algorithm can be generalized by the following expression:

$$\varphi_{\min} = \frac{\gamma_{\text{opt}}\sqrt{\pi}}{4\alpha\mathcal{Z}_2^2}\left[\frac{\gamma_{\text{opt}}\sqrt{\pi}}{4} + \sqrt{\frac{\gamma_{\text{opt}}^2\pi}{16} + 2\alpha\sigma_v^2\mathcal{Z}_2^2}\right] + \sigma_v^2, \tag{34}$$

where

$$\gamma_{\text{opt}} = \alpha\mu_{\text{opt}}\mathcal{Z}_1 + \mu_{\text{opt}}^{-1}\mathrm{Tr}(\mathbf{Q}). \tag{35}$$

IV. Simulation Results

To assess the validity of our theoretical findings, extensive simulations are carried out for different scenarios. In all the simulations, we have chosen the adaptive filter length $M = 10$ and the normalization constant $\epsilon = 10^{-6}$. In Figures 1–4, we have chosen $\sigma_q^2 = 10^{-8}$ and additive white Gaussian noise (AWGN) with a signal-to-noise ratio (SNR) of 20 dB.

For the case of real-valued data, Figures 1–2 depict the tracking behavior of the $\epsilon-$NSLMS algorithm using white and correlated Gaussian regressors, respectively. As can be seen from Figure 1, the simulation results are in a close agreement with the analytical results for values of μ up to 0.001. A closer look, into the region around $\mu = 0.001$ in Figure 1, shows that the tracking MSE possesses a minimum value of 0.01014645 at $\mu = 0.001$, which are in excellent agreement with the corresponding theoretical values of $\varphi_{\min} = 0.01012611$ and $\mu_{\text{opt}} = 0.000998$ obtained from expressions (20) and (19), respectively.

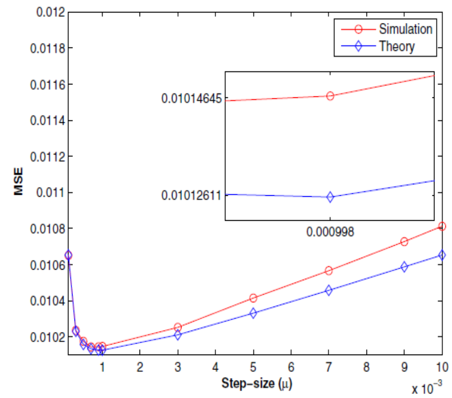

Fig. 1. Theoretical and simulated MSE of the $\epsilon-$NSLMS algorithm using white Gaussian regressors.

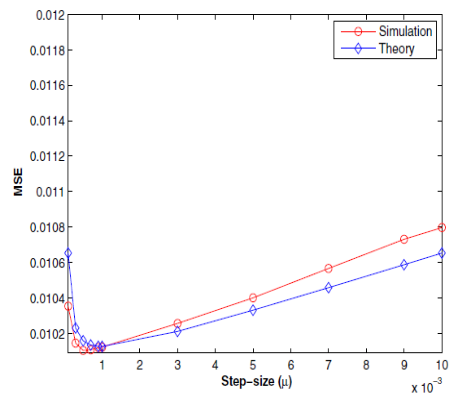

Fig. 2. Theoretical and simulated MSE of the $\epsilon-$NSLMS algorithm using correlated Gaussian regressors.

For the case of complex-valued data, Figures 3–4 demonstrate the tracking performance of the $\epsilon-$NSLMS algorithm using white and correlated Gaussian regressors, respectively. As can be seen from these figures, the simulation and analytical results are found to be in reasonable agreement for complex-valued data case.

Finally, the convergence performance of the $\epsilon-$NLMS algorithm is compared with that of the $\epsilon-$NSLMS algorithm in an unknown system identification scenario for both the real- and complex-valued data cases. Figure 5 shows the convergence comparison of the $\epsilon-$NLMS and $\epsilon-$NSLMS algorithms in an AWGN environment. On the other hand, Figure 6 shows the convergence comparison of the complex $\epsilon-$NLMS and complex $\epsilon-$NSLMS algorithms in an AWGN environment. In both these figures, the convergence curves are plotted for both white and correlated Gaussian data at an SNR of 10 dB. As can be seen from these figures, the $\epsilon-$NLMS algorithm results in superior performance over the $\epsilon-$NSLMS algorithm for the same misadjustment.

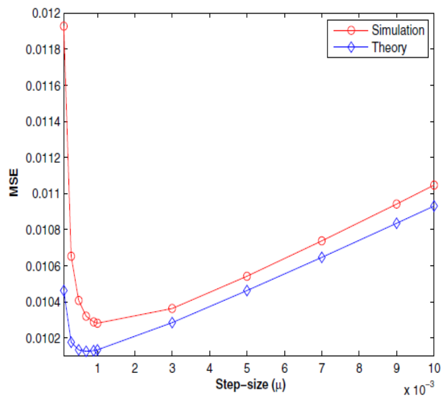

Fig. 3. Theoretical and simulated MSE of the complex $\epsilon-$NSLMS algorithm using white Gaussian regressors.

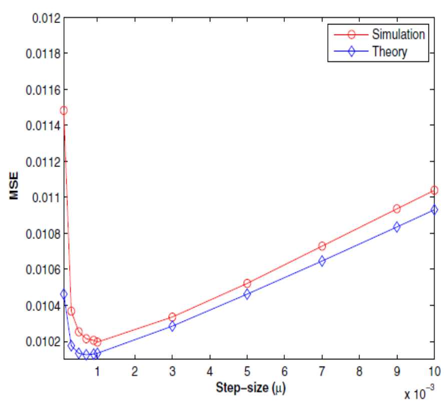

Fig. 4. Theoretical and simulated MSE of the complex $\epsilon-$NSLMS algorithm using correlated Gaussian regressors.

V. Conclusions

In this work, generalized expressions are derived for obtaining the optimal step-size and the resulting minimum MSE of the $\epsilon-$NSLMS algorithm for both cases of real- and complex-valued data. It is observed that the expressions for the optimal step-size and the corresponding optimal MSE of the $\epsilon-$NSLMS algorithm for real- and complex-valued data cases differ only by a scaling factor α. Moreover, we also observe a close match between the analytical and simulation results for real-valued data than complex-valued data. Finally, it is noted that the convergence performance of the $\epsilon-$NSLMS algorithm gets more inferior when compared with that of the $\epsilon-$NLMS algorithm when the data is complex-valued than real-valued.

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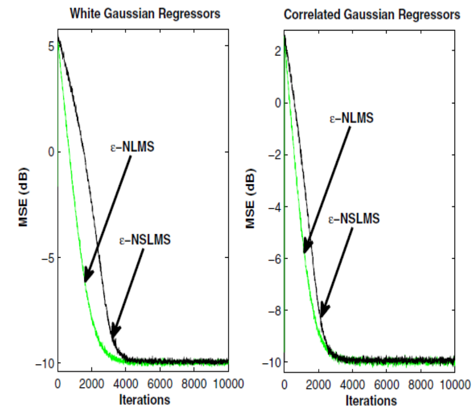

Fig. 5. Comparison of the MSE learning curves of $\epsilon-$NLMS and $\epsilon-$NSLMS algorithms in an AWGN environment.

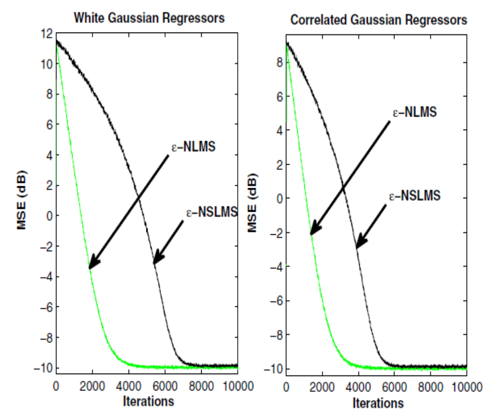

Fig. 6. Comparison of the MSE learning curves of complex $\epsilon-$NLMS and complex $\epsilon-$NSLMS algorithms in an AWGN environment.

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[9] A. Zerguine, M. Moinuddin, and S. A. A. Imam, "A noise constrained least mean fourth (NCLMF) adaptive algorithm," *Signal Processing*, vol. 91, no. 1, pp. 136–149, Jan. 2011.

7 Future Work

The Mean Square Error (MSE) performance of the four novel adaptive algorithms, which were proposed, analyzed, and evaluated in the publications listed in the author's original contributions, namely Sign Regressor Least Mean Fourth (SRLMF) [32], [33], Sign Regressor Least Mean Mixed-Norm (SRLMMN) [34], Normalized Sign Regressor Least Mean Fourth (NSRLMF) [35], [36], and Normalized Sign Regressor Least Mean Mixed-Norm (NSRLMMN) [37], [38] as well as various other novel algorithms can be tested for fixed-point arithmetic operations and subsequently compared with their respective MSE performance for floating-point arithmetic operations.

Additionally, the MSE performance of the aforementioned four algorithms as well as various other novel algorithms can be studied for the four types of quantization/loss of precision methods, namely truncate, round, round-to-zero, and convergent round.

Moreover, the fixed-point adaptive noise cancellers operating on the aforementioned four algorithms as well as various other novel algorithms can be implemented for applications such as artifacts removal from various physiological signals such as ElectroCardioGram (ECG), ElectroEncephaloGram (EEG), etc.

7.1 Contributions/Published/Accepted Manuscripts

[A1] M. M. U. Faiz and I. Kale, "A novel fixed-point leaky sign regressor algorithm based adaptive noise canceller for PLI cancellation in ECG signals," ***in Proc. of the 7th IEEE Int. Forum on Research and Technologies for Society and Industry Innovation (RTSI 2022),*** **Paris, France, pp. 186–190, Aug. 2022, DOI: https://doi.org/10.1109/RTSI55261.2022.9905081**

In this paper, a novel fixed-point Leaky Sign Regressor Algorithm (LSRA) based adaptive noise canceller has been employed for the cancellation of 60 Hz Power Line Interference (PLI) from the ECG signal. A sufficient condition for the convergence in the mean of the LSRA algorithm is also derived. The fixed-point LSRA-based adaptive noise canceller employed in this work is fully quantized using an in-house quantize function [68].

The most effective number of quantization bits required for the various parameters are found to be 6-bits and are determined through rigorous simulations. The filtered ECG signal free from 60 Hz PLI is successfully recovered using a novel 6-bit fixed-point LSRA-based adaptive noise canceller [68].

[A2] M. M. U. Faiz, S. K. Reni, and I. Kale, "A new fixed point noise cancellation method for suppressing power line interference in electrocardiogram signals," ***accepted in Proc. of the 10th IEEE Int. Conf. on E-Health and Bioengineering (EHB 2022),*** **Iasi, Romania, pp. 1–4, Nov. 2022.**

In this paper, a novel fixed-point Leaky Sign Regressor Least Mean Mixed-Norm (LSRLMMN) based adaptive noise canceller has been employed for the cancellation of 60 Hz PLI from the ECG signal. A sufficient condition for the convergence in the mean of the LSRLMMN algorithm is also derived [69].

The fixed-point LSRLMMN-based adaptive noise canceller employed in this work is fully quantized. The intention for the extensive quantization study and modeling approach was with a view to the physical integrated circuit implementation. All the modeling and simulation studies were carried out at the bit-level with various loss of precision schemes to ensure compliance with the set specification. The filter coefficients and all the data paths are quantized in order to establish at a high-level behavioral level of the parameters for a decreased complexity in integrated circuit implementation [69].

The number of quantization bits required for the primary input, secondary input, step-size, leakage factor, mixing parameter, filter coefficients, filter output, and filtered ECG signal of the fixed-point LSRLMMN-based adaptive noise canceller are found to be 8-bits for the round and convergent round methods [69].

It should be noted that at the time of making minor amendments to my thesis the publication in [A1] was available on IEEE Xplore and the manuscript in [A2] was accepted and virtually presented. Also, I was awarded the Globally Engaged Research (GER) Scholarship for the academic year 2022-2023 by the Graduate School, University of Westminster, London, United Kingdom, for the work in [A1] and [A2] at the time of making minor amendments to my thesis.

8 Conclusions

The eight adaptive algorithms analyzed and evaluated in this thesis include Sign Regressor Least Mean Square (SRLMS), Sign-Sign Least Mean Square (SSLMS), Normalized Sign Regressor Least Mean Square (NSRLMS), Normalized Sign-Error Least Mean Square (NSLMS), Sign Regressor Least Mean Fourth (SRLMF), Sign Regressor Least Mean Mixed-Norm (SRLMMN), Normalized Sign Regressor Least Mean Fourth (NSRLMF), and Normalized Sign Regressor Least Mean Mixed-Norm (NSRLMMN).

The former four aforementioned algorithms, namely SRLMS, SSLMS, NSRLMS, and NSLMS were proposed and analyzed using different methodologies by other researchers in the open literature. In this thesis, the energy conservation framework has been applied uniformly for the evaluation of the performance of various sign adaptive algorithms including SRLMS, SSLMS, NSRLMS, and NSLMS. The performance of the SSLMS, NSRLMS, and NSLMS algorithms has been analyzed and evaluated for both cases of real- and complex-valued data. While the performance of the SRLMS algorithm has been analyzed and evaluated for complex-valued data only.

The latter four aforementioned algorithms, namely SRLMF, SRLMMN, NSRLMF, and NSRLMMN are newly proposed by the author. The performance of the SRLMF algorithm has been analyzed and evaluated for both cases of real- and complex-valued data. While the performance of the SRLMMN, NSRLMF, and NSRLMMN algorithms has been analyzed and evaluated for real-valued data only.

For the case of real-valued data, new expressions for the steady-state Mean Square Error (MSE) of the SSLMS, SRLMF, SRLMMN, NSRLMS, NSLMS, NSRLMF, and NSRLMMN algorithms were derived. Moreover, new expressions for the tracking MSE of the SSLMS, SRLMF, SRLMMN, NSRLMS, NSLMS, NSRLMF, and NSRLMMN algorithms were also derived. In addition, new expressions for the optimum step-size of the SSLMS, SRLMF, NSRLMS, NSLMS, NSRLMF, and NSRLMMN algorithms were also derived. Also, a sufficient condition for the convergence in the mean of the SRLMF, SRLMMN, NSRLMF, and NSRLMMN algorithms were newly derived (see Appendix D).

For the case of complex-valued data, new expressions for the steady-state MSE of the SRLMS, SSLMS, SRLMF, NSRLMS, and NSLMS algorithms were derived. Moreover, new expressions for the tracking MSE of the SRLMS, SSLMS, SRLMF, NSRLMS, and NSLMS algorithms were also derived. Also, new expressions for the optimum step-size of the SRLMS, SSLMS, SRLMF, and NSLMS algorithms were derived (see Appendix D).

It was shown by the author that the Sign-Error Least Mean Fourth (SLMF), Sign-Sign Least Mean Fourth (SSLMF), and their variant algorithms boils down to the Sign-Error Least Mean Square (SLMS), SSLMS, and their corresponding variant algorithms for both cases of real- and complex-valued data, respectively. Thus, effectively removing the misconceptions among biomedical signal processing researchers concerning the implementation of adaptive noise cancelers using the SLMF, SSLMF, and their variant algorithms.

Moreover, the SRLMMN algorithm-based control technique employed by other researchers for the control of shunt compensator for power quality improvement in the distribution system has proven itself to be highly efficient by offering fast convergence, less steady-state error, low total harmonic distortion, and less computation complexity when compared with the Recursive Least-Squares (RLS) and Variable Step-size Least Mean Square (VSLMS) algorithms.

Furthermore, the NSRLMF algorithm is successfully employed by other researchers for power quality improvement in wind-solar based distributed generation system. The NSRLMF algorithm is shown to exhibit enhanced system dynamics as compared to the LMF algorithm.

Finally, it was shown by the author that the SRLMF and SRLMMN algorithms outperforms other sign adaptive algorithms such as the SLMS, SRLMS, and SSLMS algorithms in baseline wander and motion artifacts removal from the ECG signal.

9 Appendix A

A sample MATLAB program to generate Figure 1 in [46] is shown below:

```
clc
clear all
close all

var_n = 0.1;
sqn = sqrt(var_n);

N = 12000;
L = 1000;

mu_slms = 0.001;
mu_slmf = 0.001;

M = 10;
wo = rand(M,1);

MSE1 = zeros(1,N);
MSE2 = zeros(1,N);

for k = 1:L
    input = randn(1,N);
    v = sqn*randn(1,N);
    u = zeros(1,M);
    w1 = zeros(M,1);
    w2 = zeros(M,1);
    e1 = zeros(1,N);
    e2 = zeros(1,N);
    for j=1:N
        u = [input(j) u(1:M-1)];
        d = u*wo + v(j);
        e1(j) = d - u*w1;
        e2(j) = d - u*w2;
        % The SLMS algorithm weights update equation
        w1 = w1 + mu_slms*u'*sign(e1(j));
        % The SLMF algorithm weights update equation
        w2 = w2 + mu_slmf*u'*sign((e2(j))^3);
    end
    MSE1 = MSE1 + abs(e1).^2;
    MSE2 = MSE2 + abs(e2).^2;
end

MSE_SLMS  = MSE1/L;
```

```
MSE_SLMF  = MSE2/L;
plot(smooth(smooth(smooth(10*log10(MSE_SLMS)))),'r')
hold on
plot(smooth(smooth(smooth(10*log10(MSE_SLMF)))),'b')
legend('\bf SLMS','\bf SLMF')
xlabel('\bf Iterations')
ylabel('\bf MSE (dB)')
title('\bf Real-valued Data')
axis tight
hold off
```

The MATLAB programs to generate the remaining figures in the publications listed in the author's original contributions except in [49] are made available in the below shared location. As mentioned earlier there were no simulation results reported in [49] as it was a comments article.

https://drive.google.com/drive/folders/17DUXLOHv-_K6sJ7VDMGeSLIw5I_OjMTJ?usp=sharing

10 Appendix B

An example of the unified application of the energy conservation framework [52] in the publications listed in the author's original contributions [32]–[45] is presented here.

In [34], the expression for the steady-state Mean Square Error (MSE) $\varphi_{\mathrm{SRLMMN}} = \mathrm{E}[e_i^2]$ of the Sign Regressor Least Mean Mixed-Norm (SRLMMN) algorithm was shown to be:

$$\varphi_{\mathrm{SRLMMN}} = \frac{\mu\left(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4\right)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})}{2\left(\delta+3\bar{\delta}\sigma_v^2\right)-\mu\left(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2\right)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2, \qquad \text{(B.1)}$$

where μ is the step-size, δ is the mixing parameter ranging between $0 \le \delta \le 1$, $\bar{\delta} = 1-\delta$, $\sigma_v^2 = \mathrm{E}[v_i^2]$ is the noise variance, $\xi_v^4 = \mathrm{E}[v_i^4]$ and $\xi_v^6 = \mathrm{E}[v_i^6]$ are the fourth and sixth-order moments of the noise sequence v_i, respectively, $\sigma_u^2 = \mathrm{E}[\mathbf{u}_i^2]$ is the regressor variance, and $\mathrm{Tr}(\mathbf{R})$ is the trace of the regressor covariance matrix $\mathbf{R} = \mathrm{E}[\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_i]$.

We can obtain the expressions for the steady-state MSE of the Sign Regressor Least Mean Fourth (SRLMF) and Sign Regressor Least Mean Square (SRLMS) algorithms from (B.1) by setting δ equal to 0 and 1, respectively, as shown below:

$$\varphi_{\mathrm{SRLMF}} = \frac{\sqrt{2}\mu\xi_v^6\mathrm{Tr}(\mathbf{R})}{6\sigma_v^2\sqrt{\pi\sigma_u^2}-15\sqrt{2}\mu\xi_v^4\mathrm{Tr}(\mathbf{R})} + \sigma_v^2, \qquad \text{(B.2)}$$

$$\varphi_{\mathrm{SRLMS}} = \frac{\sqrt{2}\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{2\sqrt{\pi\sigma_u^2}-\sqrt{2}\mu\mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \qquad \text{(B.3)}$$

Note that the expression for the steady-state MSE of the SRLMF algorithm in (B.2) is the same as that derived by the author in [32].

Also, in [34], the expression for the tracking MSE $\varphi'_{\mathrm{SRLMMN}}$ of the SRLMMN algorithm was shown to be:

$$\varphi'_{\mathrm{SRLMMN}} = \frac{\mu\left(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4\right)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})+\mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\left(\delta+3\bar{\delta}\sigma_v^2\right)-\mu\left(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2\right)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2, \qquad \text{(B.4)}$$

where $\mathrm{Tr}(\mathbf{Q})$ is the trace of the covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{q}_i\mathbf{q}_i^{\mathrm{T}}]$ of the noise sequence $\mathbf{q}_i$.

Similarly, we can obtain the expressions for the tracking MSE of the SRLMF and SRLMS algorithms from (B.4) by setting δ equal to 0 and 1, respectively, as shown below:

$$\varphi'_{\text{SRLMF}} = \frac{\sqrt{2}\mu\xi_v^6\text{Tr}(\mathbf{R}) + \mu^{-1}\text{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{6\sigma_v^2\sqrt{\pi\sigma_u^2} - 15\sqrt{2}\mu\xi_v^4\text{Tr}(\mathbf{R})} + \sigma_v^2, \tag{B.5}$$

$$\varphi'_{\text{SRLMS}} = \frac{\sqrt{2}\mu\sigma_v^2\text{Tr}(\mathbf{R}) + \mu^{-1}\text{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{2\sqrt{\pi\sigma_u^2} - \sqrt{2}\mu\text{Tr}(\mathbf{R})} + \sigma_v^2. \tag{B.6}$$

Note that the expression for the tracking MSE of the SRLMF algorithm in (B.5) is the same as that derived by the author in [32].

Another such example is demonstrated in [43], wherein it is shown how to obtain the expressions for the steady-state/tracking MSE of the Normalized Sign Regressor Least Mean Fourth (NSRLMF) and Normalized Sign-Error Least Mean Square (NSRLMS) algorithms from the expressions for the steady-state/tracking MSE of the Normalized Sign Regressor Least Mean Mixed-Norm (NSRLMMN) algorithm by setting δ equal to 0 and 1, respectively.

11 Appendix C

SSD 2015

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March 16-19, 2015 in Mahdia, Tunisia

BEST PAPER AWARD

This certificate is awarded to

Mohammed Mujahid Ulla Faiz

for a very significant contribution in the field of

Computers and Information Technology

entitled

Insights Into the Convergence and Steady-State Behaviors of the SLMF and its Variants

SSD'15 General Chairs

Prof. Jawhar Ghommam & Prof. Faouzi Derbel

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12 Appendix D

The expressions for the steady-state MSE, tracking MSE, optimum step-size, and step-size bound of various sign adaptive algorithms for real-valued data, which were described in Chapters 2 to 6 are shown in Tables D.1, D.2, D.3, and D.4, respectively.

Table D.1: The steady-state MSE expressions of various sign adaptive algorithms for real-valued data.

Algorithm	Steady-State MSE Expression
SSLMS [40]	$\varphi=\frac{\mu\mathrm{Tr}(\mathbf{R})}{2\sigma_u}\left[\frac{\mu\mathrm{Tr}(\mathbf{R})}{4\sigma_u}+\sqrt{\frac{\mu^2[\mathrm{Tr}(\mathbf{R})]^2}{16\sigma_u^2}+\sigma_v^2}\right]+\sigma_v^2$
SRLMF [32]	For smaller step-sizes: $\varphi=\frac{\sqrt{2}\mu\xi_v^6\mathrm{Tr}(\mathbf{R})}{6\sigma_v^2\sqrt{\pi\sigma_u^2}}+\sigma_v^2$ For larger step-sizes: $\varphi=\frac{\sqrt{2}\mu\xi_v^6\mathrm{Tr}(\mathbf{R})}{6\sigma_v^2\sqrt{\pi\sigma_u^2}-15\sqrt{2}\mu\xi_v^4\mathrm{Tr}(\mathbf{R})}+\sigma_v^2$
SRLMMN [34]	$\varphi=\frac{\mu\left(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4\right)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})}{2\left(\delta+3\bar{\delta}\sigma_v^2\right)-\mu\left(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2\right)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})}+\sigma_v^2$
NSRLMS [43]	$\varphi=\frac{\mu\sigma_v^2}{2-\mu}+\sigma_v^2$
NSLMS [44]	$\varphi=\frac{\mu}{4}\sqrt{\frac{\pi}{2}}\left[\frac{\mu}{2}\sqrt{\frac{\pi}{2}}+\sqrt{\frac{\mu^2\pi}{8}+4\sigma_v^2}\right]+\sigma_v^2$
NSRLMF [35]	$\varphi=\frac{\mu\phi_1\xi_v^6}{6\sigma_v^2\phi_2-15\mu\phi_1\xi_v^4}+\sigma_v^2$
NSRLMMN [37]	$\varphi=\frac{\mu(\delta^2\sigma_v^2+\bar{\delta}^2\xi_v^6+2\delta\bar{\delta}\xi_v^4)\phi_1}{2(\delta+3\bar{\delta}\sigma_v^2)\phi_2-\mu(\delta^2+15\bar{\delta}^2\xi_v^4+12\delta\bar{\delta}\sigma_v^2)\phi_1}+\sigma_v^2$

Table D.2: The tracking MSE expressions of various sign adaptive algorithms for real-valued data.

Algorithm	Tracking MSE Expression
SSLMS [40]	$\varphi'=\frac{\gamma}{4}\sqrt{\frac{\pi}{2}}\left[\frac{\gamma}{2}\sqrt{\frac{\pi}{2}}+\sqrt{\frac{\gamma^2\pi}{8}+4\sigma_v^2}\right]+\sigma_v^2$
SRLMF [32]	For smaller step-sizes: $\varphi'=\frac{\sqrt{2}\mu\xi_v^6\mathrm{Tr}(\mathbf{R})+\mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{6\sigma_v^2\sqrt{\pi\sigma_u^2}}+\sigma_v^2$

	For larger step-sizes: $\varphi' = \frac{\sqrt{2}\mu\xi_v^6\mathrm{Tr}(\mathbf{R})+\mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{6\sigma_v^2\sqrt{\pi\sigma_u^2}-15\sqrt{2}\mu\xi_v^4\mathrm{Tr}(\mathbf{R})} + \sigma_v^2$
SRLMMN [34]	$\varphi' = \frac{\mu\left(\delta^2\sigma_v^2 + \bar{\delta}^2\xi_v^6 + 2\delta\bar{\delta}\xi_v^4\right)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\left(\delta + 3\bar{\delta}\sigma_v^2\right) - \mu\left(\delta^2 + 15\bar{\delta}^2\xi_v^4 + 12\delta\bar{\delta}\sigma_v^2\right)\sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R})} + \sigma_v^2$
NSRLMS [43]	$\varphi' = \frac{\mu\phi_1\sigma_v^2 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\phi_2 - \mu\phi_1} + \sigma_v^2$
NSLMS [45]	$\varphi' = \frac{\gamma\sqrt{\pi}}{4\phi_2^2}\left[\frac{\gamma\sqrt{\pi}}{4} + \sqrt{\frac{\gamma^2\pi}{16} + 2\sigma_v^2\phi_2^2}\right] + \sigma_v^2$
NSRLMF [36]	$\varphi' = \frac{\mu\phi_1\xi_v^6 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{6\sigma_v^2\phi_2 - 15\mu\phi_1\xi_v^4} + \sigma_v^2$
NSRLMMN [38]	$\varphi' = \frac{\mu c\phi_1 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2a\phi_2 - \mu b\phi_1} + \sigma_v^2$

Table D.3: The optimum step-size expressions of various sign adaptive algorithms for real-valued data.

Algorithm	**Optimum Step-Size Expression**
SSLMS [40]	$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\mathrm{E}[\lvert\lvert\mathbf{u}_i\rvert\rvert_{\mathrm{H}}^2]}}$
SRLMF [32]	For smaller step-sizes: $\mu_{\mathrm{opt}} = \sqrt{\frac{\sqrt{\pi\sigma_u^2}\mathrm{Tr}(\mathbf{Q})}{\sqrt{2}\xi_v^6\mathrm{Tr}(\mathbf{R})}}$ For larger step-sizes: $\mu_{\mathrm{opt}} = \sqrt{\mathrm{Tr}(\mathbf{Q})\left[\frac{225\left(\xi_v^4\right)^2\mathrm{Tr}(\mathbf{Q})}{36\left(\sigma_v^2\right)^2\left(\xi_v^6\right)^2} + \frac{\sqrt{\pi\sigma_u^2}}{\sqrt{2}\xi_v^6\mathrm{Tr}(\mathbf{R})}\right]} - \frac{15\xi_v^4\mathrm{Tr}(\mathbf{Q})}{6\sigma_v^2\xi_v^6}$
NSRLMS [43]	$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2\phi_1}\left[1 + \frac{\phi_1\mathrm{Tr}(\mathbf{Q})}{4\sigma_v^2\phi_2^2}\right]} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2\phi_2}$
NSLMS [45]	$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\phi_1}}$
NSRLMF [36]	$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\xi_v^6}\left[\frac{25(\xi_v^4)^2\mathrm{Tr}(\mathbf{Q})}{4(\sigma_v^2)^2(\phi_2)^2\xi_v^6} + \frac{1}{\phi_1}\right]} - \frac{5\xi_v^4\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2\phi_2\xi_v^6}$
NSRLMMN [38]	$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{c\phi_1}\left[1 + \frac{b^2\phi_1\mathrm{Tr}(\mathbf{Q})}{4a^2c{\phi_2}^2}\right]} - \frac{b\mathrm{Tr}(\mathbf{Q})}{2ac\phi_2}$

Table D.4: The step-size bound expressions of various sign adaptive algorithms for real-valued data.

Algorithm	Step-Size Bound Expression
SRLMF [32]	$0 < \mu < \frac{\sqrt{2\pi\sigma_u^2}}{3\lambda_{\max}\sigma_e^2}$
SRLMMN [34]	$0 < \mu < \frac{\sqrt{2\pi\sigma_u^2}}{\lambda_{\max}(\delta + 3(1-\delta)\sigma_e^2)}$
NSRLMF [36]	$0 < \mu < \frac{2}{1 + 3\sigma_v^2}$
NSRLMMN [37]	$0 < \mu < 2\delta + \frac{2\bar{\delta}}{1 + 3\sigma_v^2}$

Moreover, the expressions for the steady-state MSE, tracking MSE, and optimum step-size of various sign adaptive algorithms for complex-valued data, which were described in Chapters 2 and 6 are shown in Tables D.5, D.6, and D.7, respectively.

Table D.5: The steady-state MSE expressions of various sign adaptive algorithms for complex-valued data.

Algorithm	Steady-State MSE Expression
SRLMS [39]	$\varphi = \frac{2\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2} - 2\mu\mathrm{Tr}(\mathbf{R})} + \sigma_v^2$
SSLMS [41]	$\varphi = \frac{2\mu\mathrm{Tr}(\mathbf{R})}{\sigma_u^2}\left[\mu\mathrm{Tr}(\mathbf{R}) + \sqrt{\mu^2[\mathrm{Tr}(\mathbf{R})]^2 + \sigma_u^2\sigma_v^2}\right] + \sigma_v^2$
SRLMF [33]	For smaller step-sizes: $\varphi = \frac{\mu\xi_v^6\mathrm{Tr}(\mathbf{R})}{\sigma_v^2\sqrt{\pi\sigma_u^2}} + \sigma_v^2$ For larger step-sizes: $\varphi = \frac{\mu\xi_v^6\mathrm{Tr}(\mathbf{R})}{\sigma_v^2\sqrt{\pi\sigma_u^2 - 9\mu\xi_v^4\mathrm{Tr}(\mathbf{R})}} + \sigma_v^2$
NSRLMS [42]	$\varphi = \frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})}{(2-\mu)\sqrt{\pi\sigma_u^2}}\mathrm{E}\left[\frac{1}{\|\|\mathbf{u}_i\|\|_{\mathrm{H}}^2}\right] + \sigma_v^2$
NSLMS [44]	$\varphi = \frac{\mu\sqrt{\pi}}{4}\left[\frac{\mu\sqrt{\pi}}{2} + \sqrt{\frac{\mu^2\pi}{4} + 4\sigma_v^2}\right] + \sigma_v^2$

Table D.6: The tracking MSE expressions of various sign adaptive algorithms for complex-valued data.

Algorithm	Tracking MSE Expression
SRLMS [39]	$\varphi' = \frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{2\sqrt{\pi\sigma_u^2} - 4\mu\mathrm{Tr}(\mathbf{R})} + \sigma_v^2$

SSLMS [41]	$\varphi' = \frac{\gamma\sqrt{\pi}}{32}\left[\gamma\sqrt{\pi} + \sqrt{\gamma^2\pi + 64\sigma_v^2}\right] + \sigma_v^2$
SRLMF [33]	For smaller step-sizes: $\varphi' = \frac{4\mu\xi_v^6\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{4\sigma_v^2\sqrt{\pi\sigma_u^2}} + \sigma_v^2$ For larger step-sizes: $\varphi' = \frac{4\mu\xi_v^6\mathrm{Tr}(\mathbf{R}) + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\sqrt{\pi\sigma_u^2}}{4\sigma_v^2\sqrt{\pi\sigma_u^2 - 36\mu\xi_v^4\mathrm{Tr}(\mathbf{R})}} + \sigma_v^2$
NSRLMS [42]	$\varphi' = \frac{4\mathrm{Tr}(\mathbf{R})}{(2-\mu)\sqrt{\pi\sigma_u^2}}\left[\mu\sigma_v^2\mathrm{E}\left[\frac{1}{\vert\vert\mathbf{u}_i\vert\vert_{\mathrm{H}}^2}\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q})\right] + \sigma_v^2$
NSLMS [45]	$\varphi' = \frac{\gamma\sqrt{\pi}}{8\phi_2^2}\left[\frac{\gamma\sqrt{\pi}}{4} + \sqrt{\frac{\gamma^2\pi}{16} + 4\sigma_v^2\phi_2^2}\right] + \sigma_v^2$

Table D.7: The optimum step-size expressions of various sign adaptive algorithms for complex-valued data.

Algorithm	Optimum Step-Size Expression
SRLMS [39]	$\mu_{\mathrm{opt}} = \frac{1}{2}\sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2}\left[\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2} + \frac{\sqrt{\pi\sigma_u^2}}{\mathrm{Tr}(\mathbf{R})}\right]} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2}$
SSLMS [41]	$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{2\mathrm{E}[\vert\vert\mathbf{u}_i\vert\vert_{\mathrm{H}}^2]}}$
SRLMF [33]	For smaller step-sizes: $\mu_{\mathrm{opt}} = \sqrt{\frac{\sqrt{\pi\sigma_u^2}\mathrm{Tr}(\mathbf{Q})}{4\xi_v^6\mathrm{Tr}(\mathbf{R})}}$ For larger step-sizes: $\mu_{\mathrm{opt}} = \sqrt{\mathrm{Tr}(\mathbf{Q})\left[\frac{81\left(\xi_v^4\right)^2\mathrm{Tr}(\mathbf{Q})}{16\left(\sigma_v^2\right)^2\left(\xi_v^6\right)^2} + \frac{\sqrt{\pi\sigma_u^2}}{4\xi_v^6\mathrm{Tr}(\mathbf{R})}\right]} - \frac{9\xi_v^4\mathrm{Tr}(\mathbf{Q})}{4\sigma_v^2\xi_v^6}$
NSLMS [45]	$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{2\phi_1}}$

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