Problem dimensionality reduction in design of optimal IIR filters.

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PROBLEM DIMENSIONALITY REDUCTION IN DESIGN OF OPTIMAL IIR FILTERS

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ABSTRACT

A typical design of a digital IIR filter that is optimal in the sense of the weighted least squares criterion is performed by using a numerical optimisation procedure capable of searching for local minima of highly non-linear functions of vector arguments. Normally such a search is carried out inside a multidimensional space of filter parameters or, if constraints are imposed, inside a subset of the space. The dimensionality of the space is therefore equal to the number of tuneable parameters of the filter. In this article we show that this approach to designing WLS optimal filters can be effectively changed in such a way that the number of dimensions of the search space is significantly smaller than the number of tuneable coefficients. The proposed modification not only reduces the dimensionality of the filter design problem. It can also improve robustness of the design procedure and reduce errors by delivering more accurate approximations of the local minima.

1. INTRODUCTION

The problem of designing a filter that is optimal in the sense of the Weighted Least Squares criterion can be formulated as follows. Let $H(\nu)$ be a complex-valued, hermitian-symmetric function representing target frequency response of the filter and

$$G(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_{n_b}z^{-n_b}}{1+a_1z^{-1}+\cdots+a_{n_a}z^{-n_a}} \qquad (1)$$

be the prototype transfer function of the filter. Determine the vector $\mathbf{x}=[b_0,\ \cdots,\ b_{n_b},\ a_1,\ \cdots,\ a_{n_a}]$ of filter coefficients such that the following WLS cost is minimised

$$J=\int_{-0.5}^{0.5}\left|G(\exp(j2\pi\nu))-H(\nu)\right|^2W(\nu)d\nu \qquad (2)$$

where $W(\nu)$ is a real-valued, even and non-negative weight function. For obvious reason a restriction is imposed that the designed filter must be stable.

The above formulation suggests that designing an IIR filter is equivalent to solving an optimisation problem with constraints in n_a+n_b+1 dimensional space. We will show that in fact we can significantly reduce the dimensionality of the search space and confine it to only n_a dimensions.

Before revealing details of the proposed approach we briefly review the methods of stabilising the designed filter. An obvious way is to represent the denominator of (1) as a product of second-order polynomials [1]. The filter is stable if the coefficients of each factor satisfy a set of linear inequalities describing the famous "triangle of stability". Another method, proposed in [2], applies linear constraints directly to the coefficients of $A(z)$. The method is over-restrictive since it excludes some stable filters from the set of feasible solutions. Two other approaches use non-linear transformation of the filter parameters [3], and expansion of cost (2) [4]. Both modifications facilitate search of the stable optimal filter by solving an optimisation problem without constraints. Finally, it has been shown in [5] that stability of the filter can be enforced by demanding that a matrix, related to CCF state-space representation of the filter is positive definite.

Numerical methods that minimise (2) can often benefit from the access to the gradient and Hessian of the minimised cost. Some methods use them to establish directions of the search of optimal solution. Other methods need the derivatives of the cost to build local approximations of (2) as second order polynomials.

The method of reducing the dimensionality of the search space proposed here is suitable for practically all approaches reported in the research literature. It can cope with all listed above methods of enforcing stability. It also allows to effectively calculate the gradient and the Hessian of the cost.

2. FIXED-DENOMINATOR OPTIMAL FILTERS

Design of WLS optimal filters is pretty straightforward if the designed filter has an FIR structure. The coefficients of the optimal filter are the solution to a set of linear equations. The designers do not have to be concerned about multiple local minima, filter stability etc. Numerical tools needed to design filters are much simpler than in IIR case. The whole simplicity of the FIR design can be extended almost immediately to IIR case if the design is performed in two stages. First we select arbitrarily the denominator of (1) then we use optimisation methods to calculate the coefficients of the numerator that minimise (2). Of course, unless the initially chosen denominator is optimal, IIR filters designed using this approach do not minimise cost (2) as much as it is possible when full optimisation is performed. However, experienced designers are capable of placing the poles of the filter in much more useful positions than, as suggested by FIR approach, at the origin and achieve IIR filters of significantly lower order than similar quality FIR ones. To illustrate the above concept we introduce two vectors:

$$\mathbf{s}(\nu) = \frac{1}{A(e^{j2\pi\nu})}\left[1,\ e^{-j2\pi\nu},\ \cdots,\ e^{-j2\pi n_b \nu}\right]^T \tag{3}$$

and

$$\mathbf{b} = [b_0,\ b_1,\ \cdots,\ b_{n_b}]^T . \tag{4}$$

Now we can rewrite (2) as

$$J = \int_{-0.5}^{0.5} \left(\mathbf{b}^T\mathbf{s}(\nu) - H(\nu)\right)\left(\mathbf{b}^T\mathbf{s}(\nu) - H(\nu)\right)^* W(\nu)d\nu \tag{5}$$

The above expression is a quadratic function of vector **b**

$$J = \mathbf{b}^T\mathbf{M}\mathbf{b} - 2\mathbf{b}^T\mathbf{n} + r \tag{6}$$

where **M** is a symmetric, real-valued matrix

$$\mathbf{M} = \int_{-0.5}^{0.5} \mathbf{s}(\nu)\mathbf{s}^H(\nu)W(\nu)d\nu , \tag{7}$$

n is a real-valued vector

$$\mathbf{n} = \int_{-0.5}^{0.5} \mathbf{s}(\nu)H^*(\nu)W(\nu)d\nu , \tag{8}$$

and r is a scalar

$$r = \int_{-0.5}^{0.5} |H(\nu)|^2 W(\nu)d\nu . \tag{9}$$

Minimisation of (6) with respect to **b** is simple. The optimal vector of coefficients is

$$\mathbf{b}_{opt} = \mathbf{M}^{-1}\mathbf{n} \tag{10}$$

In formulas (3)-(8) superscript "T" denotes transposition, "*" complex conjugation and "H" Hermitian transposition (superposition of "T" and "*"). If needed, the integrals (7) and (8) can be quickly and accurately approximated by inverse FFT similarly to what was done in [4]. Note that the value of $\mathbf{b}_{opt}$ changes every time we change $A(z)$.

3. WLS OPTIMAL IIR FILTERS

The results derived in Section 2 will be now used to reduce dimensionality of the original filter design problem. Let us substitute $\mathbf{b}_{opt}$ defined by (10) for **b** in (6). We get

$$J = r - \mathbf{n}^T\mathbf{M}^{-1}\mathbf{n} . \tag{11}$$

It is clear now that the cost (11) does not depend on **b**. The only parameters of the filter that directly affect the cost are the coefficients of the denominator. Therefore the filter design problem is now reduced to minimisation of (11) with respect to vector $\mathbf{a} = \left[a_1,\ \cdots,\ a_{n_a}\right]^T$. By solving this problem we get the coefficients of the denominator of the transfer function $G(z)$. Once the optimal value of **a** is found then **b** can be obtained from (10).

4. GRADIENT AND HESSIAN OF THE COST

As we mentioned earlier most of the numerical optimisation procedures that are used in filter design problem require access to the gradient and Hessian of the cost. Here we show how these derivatives of (11) can be calculated. We try to keep our analysis as generic as possible. Therefore at this stage we do not assume that the parameterisation of polynomial $A(z)$ is such as shown in (1). For the purpose of ensuring filter stability other representations are often used. Examples of various parameterisations include [1]

$$A(z) = \prod_{i=1}^{n_a/2} (1 + \bar{a}_{1i}z^{-1} + \bar{a}_{2i}z^{-2}) \tag{12}$$

and [2]

$$A(z) = \prod_{i=1}^{n_a/2} (1 + \left(1 + \sin(\bar{\bar{a}}_{1i})\sin(\bar{\bar{a}}_{2i})\right)z^{-1} + \sin(\bar{\bar{a}}_{2i})z^{-2}) . \tag{13}$$

If n_a is odd then (12) and (13) have to be slightly modified to include first order factors. To reflect our flexibility in representing $A(z)$ we assume that the vector of tuneable coefficients of $A(z)$ is just $\hat{\mathbf{a}} = \left[\alpha_1,\ \cdots,\ \alpha_{n_a}\right]^T$. The relations between α_i and $A(z)$ depend on chosen parameterisation.

Now we can calculate the derivatives of the cost (11).

$$\frac{\partial J}{\partial \alpha_i} = \mathbf{n}^T\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \alpha_i}\mathbf{M}^{-1}\mathbf{n} - 2\left(\frac{\partial \mathbf{n}}{\partial \alpha_i}\right)^T \mathbf{M}^{-1}\mathbf{n} \tag{14}$$

Moreover:

$$\frac{\partial \mathbf{M}}{\partial \alpha_i} = \int_{-0.5}^{0.5} \frac{\partial \mathbf{s}(\nu)}{\partial \alpha_i}\mathbf{s}^H(\nu)W(\nu)d\nu + \int_{-0.5}^{0.5} \mathbf{s}(\nu)\frac{\partial \mathbf{s}^H(\nu)}{\partial \alpha_i}W(\nu)d\nu \tag{15}$$

$$\frac{\partial \mathbf{n}}{\partial \alpha_i} = \int_{-0.5}^{0.5} \frac{\partial \mathbf{s}(\nu)}{\partial \alpha_i}H^*(\nu)W(\nu)d\nu \tag{16}$$

and

$$\frac{\partial \mathbf{s}(\nu)}{\partial \alpha_i} = -\frac{\partial A(e^{j2\pi\nu})/\partial \alpha_i}{A^2(e^{j2\pi\nu})}\left[1,\ e^{-j2\pi\nu},\ \cdots,\ e^{-j2\pi n_b \nu}\right]^T \tag{17a}$$

$$\frac{\partial \mathbf{s}^H(\nu)}{\partial \alpha_i} = -\frac{\partial A^*(e^{j2\pi\nu})/\partial \alpha_i}{A^{*2}(e^{j2\pi\nu})}\left[1,\ e^{j2\pi\nu},\ \cdots,\ e^{j2\pi n_b \nu}\right] \quad (17b)$$

Normally, the derivatives $\partial A(e^{j2\pi\nu})/\partial \alpha_i$ that are required in (17) can be easily calculated once the parameterization of the denominator of the filter's transfer function is known. The gradient of the cost is thus a vector of length n_a having the form

$$\nabla J = \left[\frac{\partial J}{\partial \alpha_1},\ \cdots,\ \frac{\partial J}{\partial \alpha_{n_a}}\right] \quad (18)$$

The components of the vector can be calculated using (14)-(17). Once again it is recommended to use efficient numerical methods based on inverse FFT described in [4] to approximate the values of integrals (15) and (16).

In order to construct the Hessian of cost J we need to derive second derivatives of (11). Note first that if we substitute (10) in (14) we get

$$\frac{\partial J}{\partial \alpha_i} = \mathbf{b}_{opt}^T \frac{\partial \mathbf{M}}{\partial \alpha_i} \mathbf{b}_{opt} - 2\left(\frac{\partial \mathbf{n}}{\partial \alpha_i}\right)^T \mathbf{b}_{opt} \quad (19)$$

Therefore

$$\frac{\partial^2 J}{\partial \alpha_i \partial \alpha_k} = 2\frac{\partial \mathbf{b}_{opt}^T}{\partial \alpha_k}\frac{\partial \mathbf{M}}{\partial \alpha_i}\mathbf{b}_{opt} + \mathbf{b}_{opt}^T \frac{\partial^2 \mathbf{M}}{\partial \alpha_i \partial \alpha_k}\mathbf{b}_{opt} - 2\left(\frac{\partial^2 \mathbf{n}}{\partial \alpha_i \partial \alpha_k}\right)^T \mathbf{b}_{opt} - 2\left(\frac{\partial \mathbf{n}}{\partial \alpha_i}\right)^T \frac{\partial \mathbf{b}_{opt}}{\partial \alpha_k} \quad (20)$$

It is not difficult to check that:

$$\frac{\partial \mathbf{b}_{opt}}{\partial \alpha_k} = -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \alpha_k}\mathbf{b}_{opt} + \mathbf{M}^{-1}\frac{\partial \mathbf{n}}{\partial \alpha_k} \quad (21)$$

$$\frac{\partial^2 \mathbf{M}}{\partial \alpha_i \partial \alpha_k} = \int_{-0.5}^{0.5} \frac{\partial^2 \mathbf{s}(\nu)}{\partial \alpha_i \partial \alpha_k}\mathbf{s}^H(\nu)W(\nu)d\nu + \int_{-0.5}^{0.5} \frac{\partial \mathbf{s}(\nu)}{\partial \alpha_i}\frac{\partial \mathbf{s}^H(\nu)}{\partial \alpha_k}W(\nu)d\nu + \int_{-0.5}^{0.5} \frac{\partial \mathbf{s}(\nu)}{\partial \alpha_k}\frac{\partial \mathbf{s}^H(\nu)}{\partial \alpha_i}W(\nu)d\nu + \int_{-0.5}^{0.5} \mathbf{s}(\nu)\frac{\partial^2 \mathbf{s}^H(\nu)}{\partial \alpha_i \partial \alpha_k}W(\nu)d\nu \quad (22)$$

$$\frac{\partial^2 \mathbf{n}}{\partial \alpha_i \partial \alpha_k} = \int_{-0.5}^{0.5} \frac{\partial^2 \mathbf{s}(\nu)}{\partial \alpha_i \partial \alpha_k}H^*(\nu)W(\nu)d\nu \quad (23)$$

and

$$\frac{\partial^2 \mathbf{s}(\nu)}{\partial \alpha_i \partial \alpha_k} = \left(2\frac{\partial A(e^{j2\pi\nu})}{\partial \alpha_i}\frac{\partial A(e^{j2\pi\nu})}{\partial \alpha_k} - \frac{\partial^2 A(e^{j2\pi\nu})}{\partial \alpha_i \partial \alpha_k}\times A^2(e^{j2\pi\nu})\right)\left[1,\ e^{-j2\pi\nu},\ \cdots,\ e^{-j2\pi n_b \nu}\right]^T / A^4(e^{j2\pi\nu}) \quad (24)$$

The Hessian of the cost is given by

$$\nabla^2 J = \begin{bmatrix} \frac{\partial^2 J}{\partial \alpha_1 \partial \alpha_1} & \cdots & \frac{\partial^2 J}{\partial \alpha_1 \partial \alpha_{n_a}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 J}{\partial \alpha_{n_a} \partial \alpha_1} & \cdots & \frac{\partial^2 J}{\partial \alpha_{n_a} \partial \alpha_{n_a}} \end{bmatrix} \quad (25)$$

The formulas (7)-(25) show how, for a given denominator of the transfer function, we can calculate the cost, its gradient and Hessian.

5. NUMERICAL TEST

Now we are going to use the proposed approach to design a band-pass IIR filter. The target frequency response of the filter is chosen as

$$H(\nu) = \begin{cases} 0 & \text{if } |\nu| < 0.2 \text{ or } |\nu| > 0.3 \\ \exp(-j36\pi\nu) & \text{if } 0.2 < |\nu| < 0.3 \end{cases} \quad (26)$$

and the structure of the filter is described by n_a=4, n_b=30. The weight function used in the design is

$$W(\nu) = \begin{cases} 0.1 & \text{if } |\nu| < 0.213 \text{ or } |\nu| > 0.187 \\ 0.1 & \text{if } |\nu| < 0.313 \text{ or } |\nu| > 0.287 \\ 1 & \text{all other values of } \nu \end{cases} \quad (27)$$

In order to control the position of the poles we use WISE method described in [4]. The stability of the filter is achieved by linearly combining WLS cost (11) with Partial Energy of Impulse Response of filter $F(z) = 1/A(z)$.

$$J_{WISE} = (1-\lambda)(r - \mathbf{n}^T\mathbf{M}^{-1}\mathbf{n}) + \lambda\sum_{n=M}^{M+3} f^2(n) \quad (28)$$

In (28) $f(n)$ denotes impulse response of $F(z)$. It has been shown in [4] that if M and λ are properly selected then local minima of (11) placed inside the subspace of stable filters are practically identical with local minima of (28). On the other hand (28) has no local minima in the subspace of unstable filters. Therefore any optimisation method that searches for the minimum of cost (28) will end up with a stable filter that represents a local minimum of (11). In this example we selected $M = 400$ and $\lambda = 10^{-5}$. It was, however, found out that the results we obtained are very insensitive to a wide range of changes of those two parameters. Since we deploy WISE approach we do not have to perform any special parameterisation of $A(z)$. We simply use

$$A(z) = 1 + a_1 z^{-1} + \cdots + a_{n_a} z^{-n_a} \quad (29)$$

and the following expressions can be substituted in (17) and (24):

$$\frac{\partial A(\exp(j2\pi\nu))}{\partial a_i} = \exp(-ji2\pi\nu) \quad (30)$$

and

$$\frac{\partial^2 A(\exp(j2\pi\nu))}{\partial a_i \partial a_k} = 0 . \tag{31}$$

Figure 1 shows the magnitude response of the filter, while Figure 2 presents the weighted frequency response error. It would take more than 75 taps to construct an FIR filter whose quality measured by cost (11) would be not worse than that of our IIR filter.

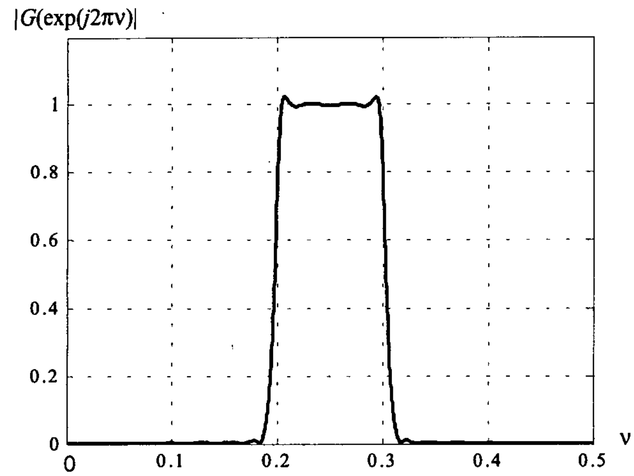

Figure 1. Magnitude response of the designed filter

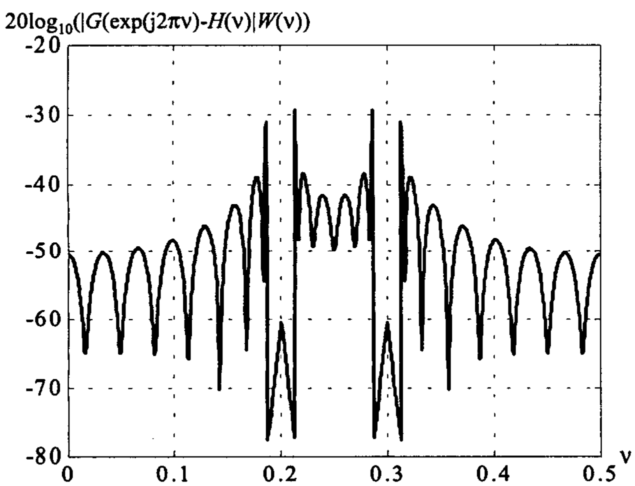

Figure 2. Magnitude of weighted frequency response error of the designed filter

5. CONCLUSIONS

The problem of designing IIR filters that are optimal in the sense of WLS criterion has been reformulated in such a way that the dimensionality of the space of optimised parameters of the filter has been substantially reduced. Local minima of the modified cost function represent exactly the same filters as the local minima of the cost that is traditionally used in filter design. The main reason that the proposed modification is recommended for use is to diminish computational effort needed to solve optimisation problem related to designing the filter.

The numerical test that is presented in the previous section of this paper shows a case where the dimensionality of the optimisation problem was reduced from 35 to only 4. This reduction was translated into 48% saving of computational time needed to design the filter. The reason that almost nine-fold reduction of problem dimensionality is only partially reflected in shortening the computational time lies partially in the fact that the cost (11) and its first and second derivatives are described by more complex mathematical formulas than those of (6).

Another advantage of the proposed approach is that the numerator of the designed filter is always tuned optimally. The numerical optimisation procedures rarely reach accurately the local minimum. Usually they stop the search in some neighbourhood of the actual solution. With the classical approach both numerator and denominator of the transfer function contain some errors. The methodology proposed here leaves us with the errors in the denominator, while the numerator of the transfer function is tuned very precisely and minimises the cost accordingly to the final choice of the denominator.

There is also an educational benefit in reformulating the filter design problem. If the designed filter has small number of non-zero poles (say $n_a \le 3$) then it is possible to calculate the cost over a dense grid of n_a-dimensional space of filter parameters and represent it in a graphical form. This may give a good information about the shape of the cost function and possible numerical problems that can be faced if one attempts minimising it. Such visualisations are without doubts very important for students. They can be also beneficial for many researchers as well.

6. REFERNCES

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