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ABSTRACT

In this paper we will estimate the term structure of daily U.K. interest rates using a range of more flexible continuous-time models. A multivariate framework is employed for the dynamic estimation and forecasting of four classic models over the eventful period of 2000-2013. The extensions are applied in two stages to four- and five-factor formulations, allowing us to assess the potential benefit of gradually increasing the model-flexibility. The Gaussian estimation methods for dynamic continuous-time models yield insightful comparative results concerning the two different segments of the yield curve, short- and long-term, respectively. In terms of in-sample performance the newly extended multi-factor general model is superior to all other restricted models. When compared to benchmark discrete-time models, the out-of-sample performance of the extended continuous-time models seems to be consistently superior with regards to the short-term segment of the yield curve.

Keywords: continuous-time models, forecasting, Gaussian estimation, multi-factor diffusion models with feedbacks, term structure of interest rates.

JEL classification: G12, G17, C51, C58

1. Introduction

Modelling and understanding the behaviour of interest rates is crucial to areas such as derivative pricing, risk management and monetary policy. In finance the key to the accuracy of any global macroeconomic model "are the yield curve models that forecast interest rates and upon which the determination of all other variables depends" (Dempster et al., 2014; p. 251).

In the aftermath of the last global financial crisis of 2007-2009, recent studies and financial regulators have suggested that the yield curve models used by market participants should allow for higher flexibility by increasing the number of factors included in the model. The Basel II Committee on Banking Supervision (2010, p12) recommended that "banks must model the yield curve using a minimum of six risk factors". The optimal number of factors to be included in the model is still open to debate. By assuming multiple sources of uncertainty, multi-factor models are more realistic and therefore able to capture the dynamics of the term structure of interest rates (hereafter TSIR) better. The overwhelming empirical literature on the TSIR mostly offers applications of two- and three-factor specifications, and only a few studies (including Duffee, 2011; Filipovic et al., 2014) have considered testing four or five-factor models. Following a principal component analysis (PCA), Steeley (2014) identified the change in the volatility as an important fourth factor, responsible for changes in the shape of the yield curve.

The empirical investigation conducted in this paper aims to test for the benefit of richer TSIR models in terms of both fitting the historical data and the forecasting performance. Following Nowman (2003, 2006) we gradually extend the general Chan, Karolyi, Longstaff and Sanders (1992) (CKLS) model to four- and five-factors. Also, we comparatively consider another three classic TSIR models nested in the CKLS framework, namely the Vasicek (1977), Cox, Ingersoll and Ross (1985) (hereafter CIR) and Brennan and Schwartz (1980) (hereafter, BS) models.

In term of the estimation technique we apply the Gaussian estimation methods of continuous-time dynamic systems developed over two decades by Bergstrom (1983, 1984, 1985, 1986, 1989, 1990) to daily U.K. interest rates over the period 2000 – 2013. This method yields quasi maximum likelihood (QML) estimates and its empirical application is justified by the considerable gain in the predictive power of continuous-time models compared with less efficient methods such as 2SLS (two stage least square) and 3SLS (three stage least square) or less sophisticated models such as discrete simultaneous equation systems and vector autoregressive (VAR) models.

Another purpose of this study is the dynamic estimation and the forecasting of above models within comparative context. Although several studies (e.g. Duffee, 2002; Diebold and Li, 2006;

Matsumura et al., 2011) have been dedicated to the forecasting performance of competing term-structure models, no previous empirical work has explored the possibility of different predictive performance of the same models across different segments of the yield curve. We separately estimate the short- and the long-maturity segments of the U.K. yield curve by considering four and five cross-sectional points for each segment of the yield curve. We use the daily GBP-LIBOR rates for the short end of the curve and the daily U.K. government nominal rates for the longer than one-year maturity segment.

The empirical results from the dynamic estimation of sixteen[1] models provide the in-the-sample estimates that are subsequently used to gauge the out-of-sample performance of the models. Two elements of forecasting analysis are brought together to construct a robust forecasting comparison framework: across six different forecasting methods (four continuous-time models are compared with the first order AR(1) and vector-autoregressive VAR(1) discrete-time models) and between the two model-extensions (four- and five-factors). We find that all the classic models are rejected in terms of goodness of fit against the general CKLS multi-factor model for both short- and long-term segments of the yield curve. However, for the long-term segment the Vasicek (1977) specification that admits negative interest rates seems to compete extremely well against the CKLS model. With regards to the out-of-sample performance, the LIBOR curve is best predicted by the CKLS model which clearly outperforms the parsimonious econometric models AR(1) and VAR(1).

The structure of this paper is as follows: In Section 2 we present a brief literature review on the theoretical models of the TSIR. Section 3 presents the gradual extension of the multivariate CKLS model with feedback effects to four- and five-factors and the data sets. Section 4 reports the empirical results from the estimation of the continuous-time models. Section 5 presents the forecasting analysis and comparison between the models. Finally, the concluding remarks are summarised in Section 6.

2. Literature Review - A Taxonomy of Continuous-Time Interest Rate Models

The current modern financial literature offers a profusion of interest rate models that have evolved along distinct theoretical paths, hence the difficulty to develop a common framework in which the models could be classified into mutually exclusive categories (Gibson et al., 2010).

Depending on specific criteria, term structure models could be classified in many ways. In terms of calibration we can differentiate between two types of theoretical models: no-arbitrage

[1] We estimate eight models for each of the two extensions, four theoretical models for the short-term segment of the yield curve and the same four models for the long-term segment.

and equilibrium models[2]. While the no-arbitrage models fit exactly, the currently observed market data providing a snapshot in time of the yield curve, the equilibrium models consider the current market prices as an output that only approximates the current term structure. While the no-arbitrage and equilibrium term structure of interest rates models have strong economic appeal, more parsimonious purely statistical models such as the Nelson and Siegel (1987) and Svensson (1994) parametric models and their more recent dynamic extensions (Diebold and Li, 2006; Laurini and Hotta, 2010) prove to possess superior predictive power.

In general, the dynamics of a yield curve model is driven by the main state variable, the short rate, that can enter the model in different forms: as a state variable itself, as an affine combination of state variables, as a sum of the squares of the state variables, as an exponential of a state variable or just as a point on the forward curve. Trying to include as many TSIR models as possible James and Webber (2000) distinguish between six main categories of interest rate models: affine yield models such as Duffie and Khan (1994, 1996); whole yield curve models such as Heath, Jarrow and Morton (1992); market models such as Jamshidian (1997) and Brace, Gatarek and Musiela (1997); price kernel models like Constantinides (1992), Rogers (1997); positive models (log-r models) like Black and Karasinski (1991) and consol models such as Brennan and Schwartz (1979). More models can still be added to this impressive list of interest models; for example, most diffusion models can be jump-augmented where the resulting models accommodate for the recognition of jump existence in the dynamics of interest rates (Das, 2002; Johannes, 2004; Jiang and Yan, 2009; Kim and Wright 2014).

With so many, occasionally overlapping classes of interest rate models, it is simpler and more relevant for our empirical investigation to broadly distinguish between two main types of interest rate models that have different practical implications. On one side, factor interest rate models bring essential information based on historical data about the pattern of future rates, hence they are more suitable for dynamic econometric and forecasting analysis and implicitly for interest rate risk management. The market or yield curve models on the other side are static, describing the position of the yield curve at one point in time and involving frequent recalibration; due to their facile calibration to observed market prices they are preferred by trading desks and other practitioners, making them extremely popular for the pricing of interest rate contingent claims. However, given our purpose of dynamic estimation and forecasting, a multi-factor model specification is considered more appropriate.

It is well known that assuming single-factor models to describe the evolution of the yield curve over time is rather unrealistic and a theoretical framework implying perfect correlation

[2] No-arbitrage models include Ho and Lee (1986), Hull and White (1990), Black et al. (1990), Duffie and Kan (1994), while equilibrium models include Vasicek (1997), Cox et al.(1985)

among the bond returns across all maturities contrasts with the empirical evidence. The early empirical literature[3] explores single-factor models and provides us with mixed results about important features observed in the dynamics of interest rates such as mean-reversion and the degree of dependence of the local volatility on the level of interest rates. While some studies imply the sensitivity of the empirical results to the choice of data sets and empirical methods (see Ioannides, 2003; Lo, 2005) other developments point to the particular choice of parametric functions for the drift and the volatility. In this regard, Ait-Sahalia (1996) rejects many classic models with a linear drift, whereas the volatility expression as a function of the interest rate level is considered too simplistic. Studies such as Brenner et al. (1996) and Koedijk et al. (1997) among others argued for a more complex combination of the level-effect and a new feature (volatility clustering and high persistence) that emerged from the discrete-time generalised auto-regressive conditional heteroscedasticity (GARCH) modelling. Recognizing the stochastic nature of the volatility constituted the first intuition towards multi-factor models. Moving from single-factor to multi-factor interest rate models was mainly achieved along two approaches facilitated by the affine framework illustrated in Duffie and Kan (1994). The first approach considers an additive structure of latent factors for the short rate (e.g. Pearson and Sun, 1994; Duffie and Kan,1996; and Babbs and Nowman,1999), while the second approach presents the model in terms of the lagged short rate and other state variables (e.g. Chen,1996; Balduzzi et al., 1996; Backus et al., 2001).

It is important to note that the framework used in our empirical investigation is rather different from both of these approaches. The state variables involved are neither short rates nor can they be interpreted in terms of level, slope and curvature as in Litterman and Scheinkman (1991). Over a series of articles Nowman (2001, 2003, 2006) estimated several two- and three-factor models such as CKLS, Vasicek and CIR models, for U.K. and Japan. Initially no feedback was considered, and the two factors were the short-term and the long-term interest rates for the two factor models; Nowman (2003) introduced feedback effects in the conditional mean component of the models for Japanene interest rates. The results selected the Vasicek model as a better model compared to CIR based on the likelihood ratio test against the unrestricted CKLS model. Ait-Sahalia and Kimmel (2010) estimated all nine Dai and Singleton's (2000) canonical affine multi-factor interest rate models using U.S. treasury data and a new estimation technique for a closed form approximation of the maximum likelihood (ML) function. Based on simulated and real data, they demonstrated that the new technique produces highly accurate estimates, simultaneously reducing the computational burden due to the analytical closed form obtained.

[3] See Chan et al. (1992), Tse (1995), Dalquist (1996); Episcopos (2000), Sanford and Martin (2006) among others.

After the recent financial crisis, interest rates have decreased and remained stable to near zero level. This observation can be translated into the *collapse* of the two first factors - level and slope - into a single factor, with the former factor disappearing. Kim and Priebsch (2013) considered the shadow-rate model where the short rate is constrained to respect the zero lower bound. They found that the three-factor shadow-rate model outperforms the three-factor affine-Gaussian model, which produces larger estimated fitting errors and unrealistic long-horizon forecasts of the short rate. Recently, Filipovic et al. (2014) empirically analysed a particular specification – the linear rational square-root (LRSQ) model, suggesting that five-factors (three term structure factors and two unspanned factors) seemed to capture the dynamics of the term structure and the volatility of interest rate changes after the 2007-2009 global financial crisis well. Traditional TSIR frameworks assume, based on economic principles, that interest rates are positive, an imposition that is no more supported by recent data on both, government and commercial bond markets, where persistent negative yields occurred since 2012. In this environment one could not reject the classic Vasicek model due to its "positivity" problem. Recently, Jarrow and van Deventer (2015) presented the conceptual framework needed for the validation of the Heath, Jarrow and Morton (1992) HJM model and argued that an economically and statistically valid model should allow for negative interest rates and multiple factors.

3. Methodology and Data

3.1. The Theoretical Modelling Framework

The theoretical modelling framework is presented in terms of the CKLS specification as it nests the other models as particular cases regarding the level-effect parameter. It is of importance to emphasize that the analysis involves three distinct theoretical models. First, there is the *basic* (underlying) continuous-time model; secondly the approximate/modified continuous-time model introduced in finance by Nowman (1997) by considering local homoscedasticity, and finally the exact multivariate discrete-time model proposed by Bergstrom (1983, 1984) as a discrete analogue to the modified continuous-time model. It is the third discrete model that will be estimated using the discrete data available.

3.1.1. The Continuous-Time Multi-Factor Interest Rate Models with Feedbacks

The well-known single factor CKLS (1992) short-term interest rate model is given by the following stochastic differential equation:

$$dr(t) = [\alpha + \beta r(t)]dt + \sigma r^{\gamma}(t)dZ(t), \qquad \text{for any } t > 0 \qquad (1)$$

where $r(t)$ is the short-term interest rate; α and β are the drift and mean-reversion constant parameters; σ is the proportional linear factor for the volatility of the short-term interest rate and γ is the proportional conditional volatility exponent known as the level-effect parameter. The disturbance term $dZ(t)$ is usually defined by a Wiener process $Z(t)$, however according to Bergstrom (1983) a more realistic model should allow for a more general type of randomness. Over a series of articles, Bergstrom (1983, 1985, 1986, 1989, 1990) developed the Gaussian methods of estimating continuous-time linear stochastic differential systems based on discrete-time data.

The true continuous-time multi-factor CKLS short-term interest rate model with n factors and feedbacks can be written within the general framework[4] provided by Bergstrom (1984) as the following system of stochastic differential equations:

$$\begin{cases} dr_1(t) = [\alpha_1 + \beta_{11} r_1(t) + \beta_{12} r_2(t) + ... + \beta_{1n} r_n(t)]dt + \zeta_1(dt) \\ dr_2(t) = [\alpha_2 + \beta_{21} r_1(t) + \beta_{22} r_2(t) + ... + \beta_{2n} r_n(t)]dt + \zeta_2(dt) \\ \vdots \\ dr_n(t) = [\alpha_n + \beta_{n1} r_1(t) + \beta_{n2} r_2(t) + ... \beta_{n,n} r_n(t)]dt + \zeta_n(dt) \end{cases} \tag{2}$$

or in vector-form as:

$$dr(t) = [\alpha + \beta r(t)]dt + \zeta(dt), \qquad \text{for any } t > 0 \tag{3}$$

where $r(t) = [r_1(t), r_2(t), ..., r_n(t)]'$ is the vector of the observable variables, $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]'$ is the vector of the *drift-level* parameters, $\beta = \{\beta_{ij}\}_{1 \le i, j \le n}$ is the *feedback* matrix whose elements are assumed non-zero[5], as implied by the close relationship between interest rates of different maturities, and $\zeta = [\zeta_1, \zeta_{2,} ..., \zeta_n]'$ is a vector of correlated random measures such that $E[\zeta_i(dt)] = 0$ for all $i = 1, ..., n$ and $E[\zeta(dt) \cdot \zeta'(dt)] = (dt)\Sigma(r,t),$ where $\Sigma(r,t) = \{\sigma_{ij}\}_{1 \le i, j \le n}$ is a positive definite matrix, with $\sigma_{ii} = \sigma_i^2 r_i^{2\gamma_i}(t)$ and $\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j r_i^{\gamma_i}(t) r_j^{\gamma_j}(t)$ for any $i \neq j$, $i, j = 1, ..., n$. The parameter γ_i measures the dependence of the volatility of the interest rate r_i on

[4] This is a more general model as the innovations are random measures and therefore the system allows for more complex sources of randomness, for example a combination of both, Brownian motion and Poisson processes.
[5] Including feedbacks allows for a causal relationship among all the variables within the system.

its own level, ρ_{ij} represents the correlation coefficient between any two distinct factors $r_i(t)$ and $r_j(t)$, and σ_i is the proportional volatility factor for the conditional volatility of $r_i(t)$.

In a theoretical paper, Bergstrom (1983) demonstrated the existence of a unique solution for the true/basis continuous-time model under the assumption of constant volatility. However, in the context of interest rate modelling this is rather unrealistic. Nowman (1997) relaxed this assumption and proposed a new continuous-time model that approximates the basic model well, by modifying only the diffusion component and therefore reducing temporal aggregation bias. The volatility only changes in a discrete manner at the beginning of each observation interval and remains constant during the interval. As a result, the approximate continuous model is defined by a different/adjusted variance-covariance matrix $\Sigma^*(r,t)=\{\sigma^*_{ij}\}_{1\le i,j\le n}$, with $\sigma^*_{ii}=\sigma_i^2 r_i^{2\gamma_i}(t'-1)$ and $\sigma\sigma^*_{ij}=\rho_{ij}\sigma_i\sigma_j r_i^{\gamma_i}(t'-1) r_j^{\gamma_j}(t'-1)$, where t' is the smallest integer such that $t'-1\le t<t'$.

In our comparative analysis, the continuous-time models considered are only different in the way the various model specifications measure the level-effect by assuming certain values for the volatility exponent parameter γ. For the Vasicek model we have the vector parameter $\gamma=0$, for the CIR model $\gamma=0.5$ and for BS model $\gamma=1$. Therefore, each model will assume a specific adjusted matrix $\Sigma^*(r,t)=\{\sigma^*_{ij}\}_{1\le i,j\le 4}$ for measuring the autocorrelation in the innovations, with a special case of a time-invariant matrix for the Vasicek model.

3.1.2. The Discrete-Time Multi-Factor Interest Rate Models with Feedbacks

Bergstrom (1983,1984) demonstrated that the *basic* continuous-time model has a unique solution that satisfies the following discrete stochastic difference equation (Phillips, 1972):

$$r(t)=e^{\beta}r(t-1)+(e^{\beta}-I)\beta^{-1}\alpha+\varepsilon(t) \qquad t=1,2,\dots,T \tag{4}$$

where $r(t)=[r_i(t)]'_{1\le i\le n}$, $\varepsilon(t)=[\varepsilon_i(t)]'_{1\le i\le n}$, $\alpha=(\alpha_i)'_{1\le i\le n}$

$$e^{\beta}=I+\sum_{k=1}^{\infty}\frac{1}{k!}\beta^k \quad \text{and} \quad E[\varepsilon(t)\cdot\varepsilon'(t)]=\int_0^1 e^{r\beta}\Sigma^*(r,t)e^{r\beta'}dr=\Omega(r,t)$$

The complete vector of structural parameters is $\theta=(\alpha_i,\beta_{ij},\sigma_i,\gamma_i,\rho_{ij})_{1\le i,j\le n}$ comprising $(3n^2+5n)/2$ single-value parameters. As in Nowman (2003, 2006) the elements of θ will be estimated by maximizing the Gaussian likelihood function or equivalently minimizing the following expression $L(\theta)$ which is equal to minus twice the logarithm of the Gaussian likelihood function:

$$L(\theta) = \sum_{t=1}^{T} \log(|\Omega(r,t)|) + \sum_{t=1}^{T} \varepsilon_t' \Omega^{-1}(r,t)\varepsilon_t \tag{5}$$

There will be sixteen discrete-time analogue models to be estimated, eight specifications (four models for $n = 4$ and $n = 5$, respectively) for each segment of the yield curve.

3.2. The Data

The development of theoretical models of the TSIR involves financial instruments with homogeneous characteristics such as term to maturity and level of credit risk. Therefore, it is important to consider empirical variables that match the conceptual framework of the models proposed. Following this argument, this study employs data from the London interbank (LIB) market and the U.K. government bond market over an extensive period of time including the financial crisis of 2007-2009. From the multitude of markets functioning inside any modern financial system, the interbank and bond markets play crucial roles. Interbank markets provide a platform for central banks for monitoring their policy interest rates. Their liquidity is of paramount importance to financial intermediation efficiency (Furfine, 2002). Bond markets are indispensable to any economy, being a very important mechanism used by governments around the world to meet capital needs and to finance their public debt.

To estimate the short end of the TSIR for the U.K. we employ London Interbank Offer Rate (LIBOR) of five maturities daily: one week, one, three, six and twelve months, collected from the Datastream. The time-interval covered starts from 3rd of January 2000 to 29th of March 2013 leading to a total of 3,455 daily observations.

The dataset for the long end of the TSIR focuses on daily nominal (spot) rates of tenor one, seven, ten, fifteen and twenty-five years, spanning the period from 4th of January 2000 to 28th of March 2013 with a total of 3,346 daily observations. The spot rates provided by Bank of England (BoE) have been estimated using the variable roughness penalty (VRP) model, a spine-based technique specifically designed to obtain a smooth curve for monetary policy analysis (Anderson and Sleath, 2001). The descriptive statistics of the data are presented in Table 1.

[Insert Table 1 Here]

4. The Estimation Results

The econometric estimation of the proposed continuous-time models is conducted in two stages corresponding to the two extensions, four- and five-factor models. All the models incorporate a linear mean-reversion drift by recognising feedback effects in all directions among the factors included in the model. Another way to explain the connection between different maturity rates along the yield curve is by assuming that the stochastic components, more specifically the individual Brownian motions are correlated as defined by the covariance matrices presented in section (3.1.1). Therefore, the parameters of most interest are the level-effect vector-parameter γ, the feedback matrix β and the correlation coefficients $(\rho_{ij})_{1\le i,j\le 4(5)}$.

4.1. Estimation Results for the Four-Factor Continuous-Time Models

The QMLE estimates of the parameters are grouped in the solution-vector θ to the optimization problem of maximizing the respective objective function and are presented in the two-panel Table 2 for the LIBOR rates and in Table 3 for the U.K. nominal interest rates. The vector parameter θ has thirty-four components under the general model CKLS and thirty under any of the restricted models. The estimates are presented separately, with the drift parameters in the panel A and the diffusion parameters in the panel B. The restricted models are tested for their explanatory power against the general CKLS model using the likelihood ratio test (LR).

4.1.1. Estimation Results for the Four-Factor Continuous-Time Models – The LIBOR Curve

The estimation results are discussed with a focus on the CKLS model with its two components: drift and diffusion. Regarding the drift parameters, the results reported in Table 2 (panel A) suggest that there is weak evidence of mean reversion in the four-factor TSIR models; most of the intercept estimates α_i are statistically significant, although very close to zero. With regard to the feedback matrix, fourteen out of sixteen parameters are significant; however, there is evidence of only small feedback in multiple directions. The feedback matrix in the more general CKLS model seems to have more significant elements when compared to the restricted models. This may suggest that the increased flexibility provided by the CKLS specification by not restricting the level-effect parameter γ, may also render a more complex relationship among the factors that are captured in the drift component.

The estimates of the diffusion parameters are presented in Table 2 (panel B). The results show that the four estimates of the level-effect vector parameter γ, are all over unity, implying a strong dependence of the volatility of the interest rate changes on the level of the interest rate itself. As a result, the best nested model should be the BS model and this is confirmed by it having the second highest likelihood function value, after the CKLS model. The BS model is followed in terms of explanatory power by the CIR and then Vasicek models. The estimates for the correlation coefficients are all positive under the CKLS model for all LIBOR rates. The estimation results for the correlation coefficients indicate that the six-month and twelve-month rates are most highly correlated with the value of the correlation coefficient ρ_{34} between 0.88 and 0.93 across the models, confirming empirically the importance of these maturities in the money markets. The other pairs of highly correlated short-term interest rates are for maturities of one-month with six-month and one-week with one-month. Based on the likelihood ratio (LR) test, the validity of all the nested models is rejected at the 1% level of significance.

[Insert Tables 2A and 2B Here]

4.1.2. Estimation Results for the Four-Factor Continuous-Time Models – The U.K. Nominal Curve

In the case of the bond market data, the estimates regarding the drift components, under the CKLS are presented in Table 3 (panel A). The estimates of the intercept parameters α_i are very small, whereas the feedback matrix only has five statistically insignificant elements, hence we conclude $\beta_{12} = \beta_{24} = \beta_{34} = \beta_{41} = \beta_{44} = 0$. The estimation results regarding the diffusion parameters are rather different from the LIBOR curve results, with much lower level-effect estimates and a different correlation structure given the behaviour of the long-term end of the yield curve. As it can be seen in the Table 3 (panel B), the components of vector γ are estimated within the range (0.00004, 0.22), suggesting a much weaker sensitivity of the conditional variance with respect to the level of interest rate; and only $\gamma_1 = 0.22$ is statistically different from zero. Therefore, for longer maturities the conditional variance does not depend on the interest rate level. As expected the correlation coefficients are higher between the spot rates corresponding to the flatter end of the term structure with $\rho_{34} = 0.95$, $\rho_{23} = 0.94$ and $\rho_{24} = 0.82$. As in the case of the LIBOR data, all the restricted models are rejected against the unrestricted CKLS model. However, the order in which the nested models better explain the data is reversed with the Vasicek model first, followed by the CIR and BS model.

[Insert Tables 3a and 3b Here]

4.2. *Estimation Results for the Five-Factor Continuous-Time Models*

In the second stage of the estimation we extended the four continuous-time models (CKLS, Vasicek, CIR and BS) from four to five factors. As the fifth factor, we added the three-month LIBOR and the 10-year U.K. nominal rate time series for the short end and the long end of the yield curve, respectively. The number of parameters to be estimated increased to fifty for the CKLS model and to forty-five for the restricted models. Relatively to the four-factor specifications, the five-factor models naturally gain more explanatory power models and the ranking among the continuous-time models in terms of goodness of fit has remained unchanged.

4.2.1. Estimation Results for the Five-Factor Continuous-time Models – The LIBOR Curve

As in the case of four-factor models the feature of mean-reversion is supported by the estimation results presented in Table 4 (panel A). Under the CKLS model the drift vector parameter has mostly statistically significant components, while results for the feedback matrix β of twenty-five components produce evidence of feedback in most directions.

Regarding the correlation between the five LIBOR rates, the new factor – the three-month LIBOR rate - appears to have a very high positive correlation with the adjacent maturity rates the six-month $(\rho_{34} = 93\%)$ and one-month LIBOR rate $(\rho_{23} = 84\%)$ respectively. However, the six-month LIBOR rate seems to be the main factor along the money market spectrum, with two highest correlation coefficients $(\rho_{34} = \rho_{45} = 93\%)$. For the shortest maturities, one-week and one-month the correlation coefficient is much lower (see Table 4, panel B). In conclusion the last three factors, the three-, six- and twelve-month LIBOR rates move closely together implying that if any twists were to be existent in the term structure of interest rates over the period 2000-2013, they should have occurred outside this three-twelve month maturity zone. Out of the nested models the best fit is provided by the BS model, followed by the CIR and Vasicek models. Based on the likelihood ratio tests, all the restricted models are rejected against the CKLS model.

[Insert Tables 4a and 4b Here]

4.2.2. Estimation Results for the Five-Factor Continuous-time Models – The U.K. Spot Curve

For U.K. spot rates the five-factor models estimation results consolidate the findings of the four-factor framework. The estimates of the level-effect parameters are very close to zero implying a homoscedastic conditional variance for all the factors, see Table 5B. Out of the five level-effect parameters only $\gamma_1 = 0.19$ is statistically significant. Therefore, the Vasicek model is the most appropriate restricted model, this fact being indicated by its second highest log-likelihood function value and a close to acceptance LR statistic value. The drift coefficients $\alpha_i (i = 1,...,5)$ are all insignificant - see Table 5 (panel A), while among the elements of the feedback matrix there is evidence of highly significant feedbacks in both directions between three pairs of factors. They are the (7-year, 10-year) pair with the highest feedback coefficient from the 10-year to 7-year spot rates of $\beta_{23} = -0.11326$; (7-year, 15-year) and the (10-year, 25-year) pair with a stronger feedback coefficient from the 10-year to the 25 year of $\beta_{23} = -0.11326$.

[Insert Tables 5A and 5B Here]

The correlation coefficient estimates are all highly significant and positive with the highest values (γ_{23} and γ_{34}) being realised consistently for two pairs of maturities, 7-year with 10-year and 10-year with 15-year, respectively. This observation is consistent with the feedback results and highlights the importance of the new factor introduced in the models - the 10-year maturity spot rates, which correspond to a crucial position on the term structure of interest rates given the fact that the 10-year U.K. discount bond market is one of the most liquid of all.

5. The Forecasting Analysis

The forecasting analysis is conducted across six different forecasting methods, using two popular metrics for the evaluation of the forecasting accuracy based on 250 out-of-sample observations (02 April 2013 to 25 March 2014). The two measures are the statistical root-mean-square-error (RMSE) measure and the economic percentage-change-in-direction (CDIR) measure. Moreover, the out-of-sample performance of the competing models is formally tested using the Clark-West (2007) and Diebold-Mariano (1995) for nested and non-nested specifications, respectively. Four continuous-time models (CKLS, Vasicek, CIR and BRSC) and two benchmark discrete-time models (VAR(1) and AR(1)) are estimated based on the time series data sets described in section 2.3. The choice of the mentioned discrete-time models as benchmarks is consistent with the specification of the discrete analogue model implied by Bergstrom's methodology, where for a $k-$th order linear stochastic differential system the discrete analogue model is a $VARMA(k, k-1)$ model. The continuous-time models considered for estimation in this study correspond to the particular case of $k=1$, hence their discrete analogues are VAR(1), with the following vector-specification:

$$r(t+1) = e^{\beta} r(t) + (e^{\beta} - I)\beta^{-1}\alpha + \varepsilon(t) \tag{6}$$

Important to note is that, while in the basic continuous model the coefficients are linear in elements of the feedback matrix β, the coefficients of the discrete-time model are exponential functions of the feedback matrix β, carrying some potential causal predictive value from the other factors, which is consistent with the financial theory of correlation among interest rates of different maturities. The corresponding VAR(1) models have been estimated in Eviews by the OLS method together the univariate AR(1) models for each individual time series. Once all six types of models have been estimated for each extension, the corresponding daily optimal ex-post point forecasts are evaluated.

5.1. The Dynamic Forecasting Algorithm

Assuming parameter stability and given the property of infinite memory of the general autoregressive models, the dynamic optimal forecasts are generated by "the chain rule". Accordingly, for an AR(1) model the h-step-ahead optimal forecast is given by the intercept plus the coefficient of the one-period lagged variable multiplied by the previous ($h-1$)-step-ahead optimal forecast. The origin observation used in the forecasting analysis is the last observation r_T $(T = 3,455)$ from the in-the-sample data set. The one-step-ahead optimal forecast is defined as $f_{T,1} = E(r_{T+1|T})$, i.e. the conditional expectation of r at time $T+1$ given all the information available up to and including time T.

Therefore, by applying the conditional expectation operator to equation (24) , the one-, two- and the h-step-ahead optimal forecasts are derived as follows:

$$\begin{aligned} f_{T,1} &= E(r_{T+1|T}) = e^{\beta} r(T) + (e^{\beta} - I)\beta^{-1}\alpha \\ f_{T,2} &= E(r_{T+2|T}) = e^{\beta} f_{T,1} + (e^{\beta} - I)\beta^{-1}\alpha \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \\ f_{T,h} &= E(r_{T+h|T}) = e^{\beta} f_{T,h-1} + (e^{\beta} - I)\beta^{-1}\alpha \end{aligned} \tag{7}$$

In order to determine the forecasting accuracy of the models, the forecast errors are aggregated using various statistical and economic forecasting metrics. Over the last two decades the literature on measures of forecast error still portrays a controversial picture documenting their various limitations[6] . Moreover, choosing the right loss function is relative to the particular purpose at hand as forecasts can be used in various decision environments either by trading desks or government officials. Acknowledging the controversy around the choice for a suitable forecasting measure, this forecasting analysis employs two standard stylized statistical and economic metrics: the RMSE (root mean squared error) and the CDIR (percentage change of direction) to evaluate the accuracy of the forecasts across the models considered. To compute these metrics the following formulae have been used:

$$RMSE_i = \sqrt{\frac{1}{H_i}\sum_{t=T+1}^{T+H_i}(r^f(t) - r^a(t))^2} \tag{8}$$

$$CDIR_i = \frac{1}{H_i}\sum_{t=T+1}^{T+H_i} z_t, \quad \text{where } z_t = \begin{cases} 1, & \text{if } \quad [r^a(t) - r^a(t-1)][r^f(t) - r^a(\mathrm{t}-1)] > 0 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

$r^a(t)$ and $r^f(t)$ are the actual and the forecasted value at time *t*, respectively.

5.2. The Comparative Forecasting Results for the Four- and Five-Factor Model

The forecasting results for the four- and five-factor extensions are organised across the forecasts methods and horizons for each maturity interest rate time series of each data set. We report the values normalised to the values of the benchmark VAR(1) model. Hence, for the RMSE, a ratio lower than one indicates an improved forecast. The opposite is true for the CDIR measure, where the larger than one the reported ratio is the better the forecast.

One general positive result that can be observed from the Tables 6-7 below is that for the CKLS model that was statistically accepted as the best in terms of goodness of fit, increasing the model flexibility also resulted in an improvement of the forecasting accuracy. Moreover, the CKLS model outperformed the benchmark models especially in the LIBOR curve case.

[Insert Tables 6 and 7 Here]

For GBP-LIBOR rates the forecasting results for the four-factor models indicate that the CKLS model performs best, with a smaller RMSE for all maturities except 12-month for the five-factor specification. However, for the CDIR measure, the Vasicek forecasts are better for

[6] The forecasting measures can often become infinite or undefined given the nature of real data.

longer maturity rates of 6-month and 12-month GBP-LIBOR rates under five-factors and 1-week under four-factor.

In the case of the U.K. nominal spot rates the forecasting performance results are mixed and they also change when we move from four- to five-factors. While for the four-factor specifications the RMSE overall forecasting performance is dominated by the Vasicek and CIR models, for the five-factor models there is limited improvement due to continuous-time models. Relative to the CDIR performance measure, the five-factor forecasts indicate some improvement, but mainly for the 7-year maturity.

5.3. Statistical Significance of Out-Of-Sample Forecasts

The statistical significance of the out-of-sample forecasts can be tested formally with the Diebold-Mariano test (Diebold & Mariano, 1995). The test is carried out under quadratic error loss. We follow the approach outlined in Diebold (2015) where we compare the forecasts produced by the various models and not the models themselves[7]. Hence, we are interested in comparing the forecasts and test for significance between different series of 250 forecasts. Each series of forecasts is identified by the same name of the model used to generate the forecasts.

Diebold (2015) discussed why the Diebold-Mariano test works well when we compare the forecasts and not the models as data generating processes. If one takes into consideration models as well some corrections may provide a better insight. For nested models, one technical problem with the Diebold-Mariano test is that under the null hypothesis that the parsimonious model is assumed to generate the data and therefore the larger model, in finite samples, is contaminated in terms of estimation because of additional unnecessary parameters. Clark and West (2007) provided an adjustment for the Diebold-Mariano tests such that their test statistic had approximately zero mean under the null hypothesis. Moreover, Clark-West test is a one sided test while the Diebold-Mariano is a two sided test. We are going to employ the Clark-West test for the nested models in the CKLS family as well as for the four-factor versus five-factor models of the same specification (e.g. four-factor Vasicek versus five-factor Vasicek) and Diebold-Mariano tests for the remaining pairs of models.

[7] Another line of inquiry would be to compare the models themselves on the basis of pseudo-out-of-sample forecasts. Clark and McCracken (2001) and Clark and McCracken (2013) highlight that the distribution of the test statistic can be very different when the null hypothesis makes use of the model specification and parameter estimation uncertainty is taken into consideration. The testing based on model specification needs then to distinguish between nested versus non-nested models. The model comparison and possible model averaging is outside the scope of this paper due to space limitation and it will be the subject of future research project.

The results of these two tests are reported in Table 8 and Table 9 for the four- and five - factor models, respectively. While the results from the Clark-West test have a straightforward standard interpretation, for the Diebold-Mariano test statistic, a negative number indicates that the first series forecasts (produced by the model on the vertical column of the table) yield a significantly lower loss error than the second forecast series. The opposite interpretation is true for the positive values and significance is evidently subject to a threshold comparison with a two-sided normal test constructed appropriately.

[Insert Tables 8 and 9 Here]

First of all, all continuous-time-models forecasts dominate the benchmark forecasts generated by VAR(1), and also the AR(1) ones. For money markets, under both four-factors and five-factors the CKLS forecast dominates the Vasicek, CIR and BS forecasts. However, for spot rates, there are maturities for which the Vasicek forecasts and the CIR forecasts that dominates the other forecasts under both four-factors and five-factors, while also CKLS sometimes produces superior forecasts. In addition, there are also spot interest rate maturities for which all forecasts are equally accurate, see Table 9.

We also have tested the forecasts produced by each model under four-factors and five-factors. Evidently, AR(1) models are left out of this analysis since they are considered unidimensional. The results are depicted in Table 10. For the money market LIBORs the four-factors forecasts dominate five-factor forecasts for all models except CKLS for 1-week rate and 6-month rate that gives equally accurate forecasting results. For the spot rates, remarkably, the four-factor forecasts are better than the five-factor forecasts under all continuous-time models, while for the benchmark model VAR there is a benefit for forecasting under a five-factor specification but only for the 7-year maturity.

[Insert Table 10 Here]

6. Conclusions

Despite a voluminous literature on interest rate models, there are still several open questions regarding certain aspects of interest rate modelling. The empirical study conducted in this paper tries to bring more light over two issues. The superiority of multi-factor interest rate term structure models relative to single factor formulations is well established in the relevant literature. However, until recently, following the PCA study by Litterman and Scheinkman (1991) very little empirical work has considered models beyond three factors. Events like the financial crisis of 2007-2009 called for this threshold to be reviewed with a clear necessity of increasing the flexibility of the existent models. In line with these new recommendations four continuous-time term structure models (CKLS, Vasicek, CIR and BS) are extended to four and to five-factors, respectively, following Nowman's (2003, 2006) approach.

The empirical results of the dynamic estimation favour the five-factor models over the four-factor models, the addition of the fifth factor substantially increasing the goodness of fit. After a closer examination, the change from four to five-factor specifications suggests that the parameters measuring the dependence of volatility on the interest rates levels may be overestimated when the model is less flexible.

Another benefit of increasing the model flexibility is that one could observe the change in the structure of the variance-covariance matrix between the two extensions. This allows for a clearer identification of where the strongest connections among the factors are situated along the term structure. This feature of the analysis has important implications for investment decision making process; investors who focus on certain segments of term structure of interest rates could determine, given the structure of the estimated covariance matrix, the regions where a twist/inversion in the shape of the yield curve may occur or be absent.

Our forecasting performance analysis reveals that the CKLS forecasts overall outperforms the other forecasts for the period investigated in terms of RMSE and CDIR for LIBORs while for spot rates the Vasicek and CIR forecasts are occasionally better. An analysis based on significance of the Diebold-Mariano and Clark-West tests seems to confirm these results. The CKLS forecasts outperform the other forecasts for LIBORs, under both four and five-factor specifications. For the spot rates Vasicek and CIR forecasts are better but only for some maturities. Combining this conclusion with the result that the four-factor specification of the same model seems to give superior forecasts than the five-factor specification of the same model. We conclude that CKLS models bring superior modelling capabilities to the term structure of interest rates up to one year, while for the long-term segment simpler models like Vasicek and CIR may be more useful.

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Table 1

Summary Statistics for the LIBORs and Spot Yield Rates. This table reports the standard summary statistics for all the univariate time series of LIBOR and spot rates. Five distinct maturities are examined: one-week, one-, three-, six- and twelve-month LIBOR rates, respectively and one-, seven-, ten-, fifteen- and twenty-year government yield spot rates. The data was sampled daily over the period January 1, 2000 to March 29, 2013, from Datastream. The difference in the number of observations is the result of how the two data sources Datastream and BoE have treated the entries of interest rates corresponding to bank holidays. The Datastream has equalled the interest rates on bank holidays to the level of the previous day, while the BoE kept them as unavailable.

LIBOR- GBP			LEVEL		
Maturity	1W	1M	3M	6M	12M
Observations	3,455	3,455	3,455	3,455	3,455
Mean	3.438	3.5237	3.6636	3.7855	3.9983
Median	4.1369	4.0992	4.1891	4.39	4.5663
Maximum	6.9409	6.75	6.9038	6.7988	6.8877
Minimum	0.48	0.4913	0.5069	0.6013	0.9081
Std. Dev.	2.0715	2.0955	2.0537	1.9732	1.8462
Skewness	-0.4511	-0.4429	-0.4214	-0.3926	-0.3388
Kurtosis	1.5768	1.5958	1.6462	1.6562	1.6776
Jarque-Bera	408.76	396.81	366.07	348.70	317.86
Probability	0	0	0	0	0
U.K. Spot			LEVEL		
Maturity	1Y	7Y	10Y	15Y	25Y
Observations	3,346	3,346	3,346	3,346	3,346
Mean	3.3274	4.0256	4.208	4.3472	4.3156
Median	4.2342	4.4636	4.5004	4.5239	4.3997
Maximum	6.3652	6.1509	5.7299	5.2352	5.0466
Minimum	0.1346	0.9909	1.5889	2.2856	3.0762
Std. Dev.	2.0136	1.2198	0.9389	0.6596	0.4055
Skewness	-0.4447	-0.9565	-1.2547	-1.6127	-1.1542
Kurtosis	1.5894	2.9242	3.7589	4.8883	3.9158
Jarque-Bera	387.67	510.96	958.19	1947.49	859.88
Probability	0	0	0	0	0

Table 2

The U.K. LIBOR Curve - Estimation results for the four-factor models. This table reports the parameter estimates for the CKLS, Vasicek, CIR and BS four-factor models. Panel A presents the drift coefficients, while Panel B presents the diffusion coefficients. The variables examined are the daily one week, one-, six- twelve-months LIBOR rates over the period January 1, 2000 to March 29, 2013. The likelihood ratio test (LRT) suggests that all the nested models are rejected against the unrestricted CKLS multi-factor model as the critical values is $\chi^2(4df, 1\%) = 13.28$. The level of significance is indicated as follows: *** for 1%, ** for 5% and * for 10%.

Panel A	CKLS	Vasicek	CIR	BS
α_1	-4.60E-07***	-6.62E-04***	-0.0001***	4.16E-05***
α_2	3.05E-05***	-1.09E-04***	-0.0001***	-5.00E-07***
α_3	3.38E-05***	-2.72E-05***	-0.0001***	-1.98E-05***
α_4	4.50E-05***	3.40E-05***	-0.0001***	-4.05E-05***
β_{11}	0.0144***	-0.0175*	-0.1993***	0.0600***
β_{12}	-0.0173***	-0.0032	0.2664***	-0.0424***
β_{13}	-0.0039	-0.0860***	-0.1434***	-0.0431***
β_{14}	0.0043***	0.1155***	0.0732***	0.0238***
β_{21}	0.0216***	0.0306***	0.0160***	0.0362***
β_{22}	-0.0196***	-0.0536***	-0.0068**	-0.0348***
β_{23}	0.0015	0.0320***	-0.0378***	-0.0077***
β_{24}	-0.0039***	-0.0071*	0.0301***	0.0059***
β_{31}	-0.0029***	0.0049***	0.0171***	0.0073***
β_{32}	0.01334***	-0.0035***	-0.0043*	-0.0076***
β_{33}	-0.0094***	0.0003	-0.0447***	-0.0034**
β_{34}	-0.0015**	-0.0013	0.0336***	0.0036***
β_{41}	-0.0135***	0.0022	0.0111***	-0.0168***
β_{42}	0.0235***	-0.0073***	0.003	0.0164***
β_{43}	-0.0055***	0.0253***	-0.0413***	-0.0051***
β_{44}	-0.0053***	-0.0209***	0.0284***	0.0053***

Panel B	CKLS	Vasicek	CIR	BS
γ_1	1.5940***	0.0000	0.5000	1.0000
γ_2	1.2237***	0.0000	0.5000	1.0000
γ_3	1.0308***	0.0000	0.5000	1.0000
γ_4	1.3951***	0.0000	0.5000	1.0000
σ_1	0.2059***	0.0011***	0.0056***	0.0288***
σ_2	0.0181***	0.0003***	0.0016	0.0079***
σ_3	0.0074***	0.0003***	0.0013	0.0057***

σ_4	0.0282***	0.0003***	0.0015	0.0072***
ρ_{12}	0.5438***	0.4956***	0.5348***	0.5993***
ρ_{13}	0.2577***	0.1820***	0.2214***	0.0494***
ρ_{14}	0.2154***	0.1379***	0.1767***	-0.0988***
ρ_{23}	0.7774***	0.7387***	0.7573***	0.5311***
ρ_{24}	0.6632***	0.6037***	0.6193***	0.2644***
ρ_{34}	0.9331***	0.9275***	0.9282***	0.8795***
LogLF	112,577.66	105,903.42	109,627.13	110,947.21
LRTest		13,348.48***	5,901.06***	3,260.90***

Table 3

The U.K. SPOT Curve - Estimation results for the four-factor models. This table reports the parameter estimates for the CKLS, Vasicek, CIR and BS four-factor models. Panel A presents the drift coefficients, while Panel B presents the diffusion coefficients. The variables examined are the daily one-, seven-, fifteen- and twenty-five years U.K. Nominal spot rates over the period January 3, 2000 to March 29, 2013. The likelihood ratio test (LRT) suggests that all the nested models are rejected against the unrestricted CKLS multi-factor model as the critical values is $\chi^2(4df, 1\%) = 13.28$. The level of significance is indicated as follows: *** for 1%, ** for 5% and * for 10%.

Panel A	CKLS	Vasicek	CIR	BS
α_1	-0.000256***	-0.000182***	0.000377***	0.000003***
α_2	-0.000069***	-0.000086***	-0.000001***	0.000140***
α_3	-0.000001	0.000116***	0.000130***	-0.000036***
α_4	0.000059***	0.000285***	0.000127**8	0.000006***
β_{11}	0.003367***	0.007227***	0.003633***	0.006915***
β_{12}	-0.005046	-0.018494***	-0.021332***	-0.007371***
β_{13}	-0.014834***	0.011987**	0.045019***	-0.003830
β_{14}	0.022547***	0.003393	-0.037905***	0.005698***
β_{21}	0.008699***	0.011430***	0.008137***	0.024932***
β_{22}	-0.030251***	-0.028597***	-0.019111***	-0.073313***
β_{23}	0.022878***	0.020751***	0.006231	0.095732***
β_{24}	0.000049	-0.001351	0.004190	-0.051469***
β_{31}	0.005311***	0.007303***	0.002824**	-0.000621
β_{32}	-0.021271***	-0.016580***	-0.006591*	0.001989
β_{33}	0.015795***	0.010809***	-0.001527	-0.000609
β_{34}	0.000000	-0.003871	0.001778	-0.000532
β_{41}	0.002037**	0.005679***	-0.000089	-0.013891***
β_{42}	-0.013323***	-0.016648***	-0.001589	0.029471***
β_{43}	0.011831***	0.022238***	-0.001639	-0.011375*
β_{44}	-0.002234	-0.017888***	-0.000247	-0.005900

Panel B	CKLS	Vasicek	CIR	BS
γ_1	0.2181***	0	0.5	1
γ_2	0.0244	0	0.5	1
γ_3	4E-06	0	0.5	1
γ_4	0.0950	0	0.5	1
σ_1	0.825E-03**	0.371E-03**	0.0026*	0.0284***
σ_2	0.5E-03	0.492E-03**	0.0029**	0.0176*
σ_3	0.477E-03	0.463E-03**	0.0024	0.0091*
σ_4	0.608E-03**	0.440E-03**	0.0022**	0.0101
ρ_{12}	0.6670***	0.6392***	0.6550	0.6138***
ρ_{13}	0.5199***	0.4722***	0.5430	0.2942***
ρ_{14}	0.429***	0.3786***	0.4678	-0.0938***
ρ_{23}	0.9356***	0.9294***	0.9427	0.6793***
ρ_{24}	0.8203***	0.8074***	0.8342	0.0786***
ρ_{34}	0.9450***	0.9421***	0.9476	0.7479***
Log LF	105,776.12	105,661.29	104,941.82	100,376.30
LRTest		229.66***	1,668.60***	10,799.62***

Table 4

The U.K. LIBOR Curve - Estimation results for the five-factor models. This table reports the parameter estimates for the CKLS, Vasicek, CIR and BS four-factor models. Panel A presents the drift coefficients, while Panel B presents the diffusion coefficients. The variables examined are the daily one-week, one-, three-, six- and twelve-months LIBOR rates over the period January 1, 2000 to March 29, 2013. The likelihood ratio test (LRT) suggests that all the nested models are strongly rejected against the unrestricted CKLS multi-factor model as the critical value is $\chi^2(5df, 1\%) = 15.90$. The level of significance is indicated as follows: *** for 1%, ** for 5% and * for 10%.

Panel A	CKLS	VASICEK	CIR	BS
α_1	-0.015E-03***	-0.096E-03*	-0.099E-03***	0.135E-03***
α_2	-0.7E-05***	-0.041E-03***	0.1E-04	0.032E-03***
α_3	-0.032E-03***	-0.055E-03***	0.013E-03***	0.2E-05***
α_4	-0.012E-03***	0.5E-05	0.117E-03***	0.089E-03***
α_5	-0.014E-03***	0.088E-03***	0.290E-03***	0.17E-03***
β_{11}	-0.0204***	-0.1993***	-0.3669***	-0.1849***
β_{12}	0.0372***	0.3220***	0.6183***	0.2583***
β_{13}	-0.0606***	-0.3699***	-0.4424***	-0.0839***
β_{14}	0.0484***	0.3282***	0.1871***	0.0152
β_{15}	-0.0084***	-0.0821***	0.0003	-0.0112**
β_{21}	0.0193***	-0.0037	-0.0169***	-0.0021
β_{22}	-0.0101***	0.0175***	0.0638***	0.0174***
β_{23}	-0.0297***	-0.0481***	-0.0525***	-0.0153***
β_{24}	0.0250***	0.0475***	-0.0068	0.957E-03
β_{25}	-0.0048***	-0.0126***	0.0121***	-0.0017
β_{31}	0.0241***	0.0138***	0.0051***	-0.0006
β_{32}	-0.0167***	-0.0083**	0.0212***	0.0106***
β_{33}	0.0015***	-0.0091**	0.0093***	0.0211***
β_{34}	-0.0231***	-0.182E-03***	-0.0608***	-0.0502***
β_{35}	0.0146***	0.0048	0.0252***	0.0188***
β_{41}	0.0178***	0.0106***	0.0024	-0.0027
β_{42}	-0.0039***	-0.0075**	0.0207***	0.0133***
β_{43}	-0.0241***	0.0092***	-0.0019	-0.0140***
β_{44}	0.0096***	-0.0129***	-0.0158***	0.0219***
β_{45}	0.734E-03*	0.713E-03	-0.0071***	-0.0199***
β_{51}	0.0117***	0.0027	0.0049	0.0147***
β_{52}	0.0013	-0.0002	-0.0018	-0.0239***
β_{53}	-0.0140***	0.0325***	0.0364***	0.0340***
β_{54}	-0.0043	-0.0342***	-0.0030	-0.783E-03
β_{55}	0.0054***	-0.0022	-0.0409***	-0.0268***

Panel B	CKLS	VASICEK	CIR	BS
γ_1	1.5881***	0	0.5	1
γ_2	1.2221***	0	0.5	1
γ_3	0.8683***	0	0.5	1
γ_4	0.9292***	0	0.5	1
γ_5	1.2779***	0	0.5	1
σ_1	0.2050***	0.0012***	0.0067***	0.0283***
σ_2	0.0176***	0.0003***	0.0016***	0.0086***
σ_3	0.0040	0.255E-03***	0.0010***	0.0054***
σ_4	0.0051	0.271E-03***	0.0012***	0.0056***
σ_5	0.0185***	0.339E-03***	0.0017***	0.0079***
ρ_{12}	0.5596***	0.5466***	0.6241***	0.5806***
ρ_{13}	0.3044***	0.3226***	0.1759***	0.3599***
ρ_{14}	0.2413***	0.2430***	-0.0654***	0.2168***
ρ_{15}	0.1895***	0.1975***	-0.2249***	0.0471***
ρ_{23}	0.8398***	0.8448***	0.6959***	0.7314***
ρ_{24}	0.7520***	0.7467***	0.4051***	0.3935***
ρ_{25}	0.6212***	0.6085***	0.1229***	-0.055E-03
ρ_{34}	0.9266***	0.9303***	0.8662***	0.8042***
ρ_{35}	0.7940***	0.78430***	0.6324***	0.4563***
ρ_{45}	0.9273***	0.9284***	0.9044***	0.8447***
LogLF	145,178.45	137,767.67	141,026.26	142,591.04
LRTest		7,410.78***	4,152.19***	2,587.41***

Table 5

The U.K. Spot Curve: Estimates for the five-factor models. This table reports the parameter estimates for the CKLS, Vasicek, CIR and BS five-factor models. Panel A presents the drift coefficients, while Panel B presents the diffusion coefficients. The variables examined are the daily one-, seven-, ten-, fifteen- and twenty-five year U.K. Nominal spot rates over the period January 3, 2000 to March 29, 2013. The likelihood ratio test (LRT) suggests that all the nested models are rejected against the unrestricted CKLS multi-factor model as the critical value is $\chi^2(5df, 1\%) = 15.90$.

Panel A	CKLS	VASICEK	CIR	BS
α_1	-0.012E-03***	-0.027E-03	0.019E-03	0.139E-03***
α_2	1E-08	1E-09	1E-06	-0.025E-03***
α_3	-5E-06	-2E-06	-1E-06	-0.109E-03***
α_4	0.231E-03***	0.0002***	0.3E-03***	-0.042E-03***
α_5	0.390E-03***	0.0003***	0.651E-03***	0.029E-03**
β_{11}	0.432E-03	-0.0028***	-0.0023***	0.256E-03
β_{12}	-0.0018**	0.0014	0.0009	-0.0188***
β_{13}	0.006	0.0040	0.0026	0.0185**
β_{14}	-0.0061**	-0.0036	-0.0033	0.0082
β_{15}	0.0014	0.923E-03	0.0003	-0.0140***
β_{21}	0.0013	-0.647E-03	-0.0020**	0.0017*
β_{22}	0.0582***	0.0580***	0.0624***	0.0228***
β_{23}	-0.1133***	-0.1090***	-0.1207***	-0.0641***
β_{24}	0.0552***	0.0525***	0.06***	0.0412***
β_{25}	-0.0015	-0.46E-03	-0.0003	-0.0029
β_{31}	0.0012	-0.0012	-0.0012	0.836E-03
β_{32}	0.0257***	0.0381***	0.0262***	0.0125***
β_{33}	-0.0315***	-0.0540***	-0.0318***	-0.0268***
β_{34}	-0.0138***	0.0036	-0.0153***	-0.424E-03
β_{35}	0.0188***	0.0141***	0.0223***	0.0152***
β_{41}	0.093E-03	-0.0019**	-0.0011	8E-06
β_{42}	0.0101***	0.0124***	0.0094	0.686E-03***
β_{43}	0.287E-03	-0.024E-03	0.046E-03	-0.647E-03
β_{44}	-0.0229***	-0.0224***	-0.0239***	-0.0130*
β_{45}	0.0071***	0.0083*	0.0086	0.0128***
β_{51}	0.0012	-8E-05	-0.392E-03	0.011E-03
β_{52}	-0.0197***	-0.0175***	-0.0241***	-0.0100**
β_{53}	0.0462***	0.0405***	0.0565***	0.0076
β_{54}	-0.0318***	-0.0232**	-0.0357***	0.0028
β_{55}	-0.0052**	-0.0077	-0.0118	-0.0024

Panel B	CKLS	VASICEK	CIR	BS
γ_1	0.199***	0	0.5	1
γ_2	1E-06***	0	0.5	1
γ_3	1.57E-04	0	0.5	1
γ_4	0.0095	0	0.5	1
γ_5	0.0409	0	0.5	1
σ_1	0.739E-03***	0.368E-03***	0.0025***	0.0263***
σ_2	5.12E-04	0.460E-03***	0.0027***	0.0173***
σ_3	5.23E-04	0.461E-03***	0.0026***	0.0144***
σ_4	5.10E-04	0.435E-03***	0.0023***	0.0118***
σ_5	5.30E-04	0.423E-03***	0.0022***	0.0106
ρ_{12}	0.5665***	0.5732***	0.5815***	0.5488***
ρ_{13}	0.4695***	0.4531***	0.4994***	0.4899***
ρ_{14}	0.4055***	0.3448***	0.4380***	0.4447***
ρ_{15}	0.3319***	0.2153***	0.3546***	0.3682***
ρ_{23}	0.9831***	0.9762***	0.9821***	0.9802***
ρ_{24}	0.9396***	0.9033***	0.9301***	0.9200***
ρ_{25}	0.8344***	0.7393***	0.8073***	0.7794***
ρ_{34}	0.9780***	0.9640***	0.9738***	0.97061**
ρ_{35}	0.8814***	0.8176***	0.8636***	0.8459***
ρ_{45}	0.9491***	0.9269***	0.9436***	0.9368***
LogLF	138,495.64	138,482.50	137,511.71	134,136.69
LRTest		13.14	983.93	4,358.95

The level of significance is indicated as follows: *** for 1%, ** for 5% and * for 10%.

Table 6

The Forecasting Comparison Results for the LIBOR rates. This table reports the out-of-sample performance of the four- and five-factor continuous-time models versus the two discrete models VAR(1) and AR(1). RMSE is the root mean squared error and CDIR is the percentage change in direction accuracy measures calculated over the period 2 April 2013 to 25 March 2014. The results are presented for each individual time series (one-week, one-, six- and twelve-month maturity rates) as a ratio versus the benchmark taken as the VAR(1) model.

GBP LIBOR	CKLS	VASICEK	CIR	BS	AR(1)	VAR(1)
4 factors			RMSE250			
1W	**0.11**	2.99	0.74	0.91	0.34	1.00
1M	**0.31**	2.37	0.40	1.48	0.39	1.00
6M	**0.60**	1.13	0.78	0.63	0.76	1.00
12M	1.06	1.16	**0.98**	1.11	0.63	1.00
4 factors			CDIR250			
1W	1.58	**1.71**	1.00	1.00	1.00	1.00
1M	0.73	1.00	1.00	0.73	1.00	1.00
6M	0.80	1.00	1.00	1.00	1.00	1.00
12M	0.81	1.00	0.99	1.00	1.00	1.00
GBP LIBOR	CKLS	VASICEK	CIR	BS	AR(1)	VAR(1)
5 factors			RMSE250			
1W	**0.31**	2.00	0.48	1.08	0.42	1.00
1M	**0.43**	1.46	1.29	0.57	0.52	1.00
3M	**0.26**	1.12	1.24	0.53	0.56	1.00
6M	**0.95**	1.21	10.99	18.53	1.37	1.00
12M	1.92	0.96	11.86	5.81	**0.65**	1.00
5 factors			CDIR250			
1W	**1.71**	1.71	1.23	1.00	1.00	1.00
1M	1.00	1.00	0.73	0.73	1.00	1.00
3M	**1.04**	1.00	1.04	1.00	1.00	1.00
6M	0.97	**1.21**	0.97	0.97	1.21	1.00
12M	1.30	**1.38**	1.04	1.04	1.30	1.00

Table 7

The Forecasting Comparison Results for the U.K. Spot Rates. This table reports the out-of-sample performance of the four- and five-factor continuous-time models versus the two discrete models VAR(1) and AR(1). RMSE is the root mean squared error and CDIR is the percentage change in direction accuracy measures calculated over the period 2 April 2013 to 25 March 2014. The results are presented for each individual time series (one-year,7-year, 15-year and 25-year maturities) as a ratio versus the benchmark taken as the VAR(1) model.

U.K. SPOT	CKLS	VASICEK	CIR	BS	AR(1)	VAR(1)
4 factors			RMSE250			
1Y	1.01	1.02	**0.99**	1.03	1.00	1.00
7Y	**0.99**	1.00	1.00	1.03	18.07	1.00
15Y	1.00	**0.99**	**0.99**	1.02	2.09	1.00
25Y	1.02	**0.99**	**0.99**	1.04	9.82	1.00
4 factors			CDIR250			
1Y	1.02	1.01	**1.08**	0.93	1.09	1.00
7Y	0.99	0.99	0.99	0.94	0.94	1.00
15Y	1.00	**1.16**	1.12	1.07	1.00	1.00
25Y	1.01	**1.02**	1.01	0.96	1.01	1.00
U.K. SPOT	CKLS	VASICEK	CIR	BS	AR(1)	VAR(1)
5 factors			RMSE250			
1Y	1.01	1.00	**0.99**	**0.99**	**0.99**	1.00
7Y	1.03	1.01	1.00	1.01	18.10	1.00
10Y	1.03	1.01	1.00	1.01	1.01	1.00
15Y	1.01	1.00	1.00	1.00	2.09	1.00
25Y	**0.99**	1.00	1.01	**0.99**	9.82	1.00
5 factors			CDIR250			
1Y	0.98	1.11	**1.16**	1.05	**1.16**	1.00
7Y	0.97	**1.02**	**1.02**	**1.02**	0.95	1.00
10Y	0.93	0.96	0.93	0.96	0.91	1.00
15Y	1.07	1.06	0.99	**1.11**	0.98	1.00
25Y	**1.11**	1.00	1.00	1.00	0.99	1.00

Table 8

Diebold-Mariano and Clark-West tests results for the forecasts generated from four-factor continuous-time models and VAR(1) and AR(1) models. This table reports the values of the Diebold-Mariano tests for all pairs of models and all interest rate maturities. The Clark-West results for the nested continuous-time models are entered in italic font. The models in bold produced the best forecasting results over the period 2 April 2013 to 25 March 2014.

LIBOR						Spot					
1W	Vasicek	CIR	BS	VAR	AR1	1Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*190.39*	*65.90*	*77.40*	-53.84	-20.04	CKLS	*-14.60*	*11.12*	*17.43*	-2.23	-2.23
Vasicek		124.18	117.22	-53.78	-19.88	**Vasicek**		-9.58	-18.66	-2.27	-2.26
CIR			-38.89	-53.84	-20.03	CIR			5.78	-2.24	-2.23
BS				-53.84	-20.03	BS				-2.27	-2.26
VAR					57.15	VAR					0.37
1M	Vasicek	CIR	BS	VAR	AR1	7Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*197.81*	*28.42*	*135.02*	-47.31	-21.93	**CKLS**	*19.21*	*19.91*	*19.78*	-8.37	-26.84
Vasicek		136.36	50.95	-47.29	-21.86	Vasicek		-16.87	-20.56	-8.60	-26.84
CIR			-56.52	-47.31	-21.93	CIR			-20.99	-8.55	-26.84
BS				-47.30	-21.89	BS				-8.07	-26.84
VAR					53.61	VAR					-26.76
6M	Vasicek	CIR	BS	VAR	AR1	15Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*30.56*	*15.50*	*10.87*	-12.93	-9.79	CKLS	*-5.85*	*5.85*	*21.73*	-9.33	-15.69
Vasicek		21.10	20.70	-12.93	-9.79	Vasicek		1.33	-23.82	-10.08	-15.99
CIR			11.41	-12.93	-9.79	**CIR**			-23.60	-10.09	-15.99
BS				-12.93	-9.79	BS				-8.92	-15.47
VAR					17.18	VAR					-12.20
12M	Vasicek	CIR	BS	VAR	AR1	25Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*11.61*	*3.56*	*10.05*	-8.05	-9.79	CKLS	*-16.82*	*-15.32*	*23.16*	-7.38	-29.97
Vasicek		6.22	7.68	-8.05	-9.79	**Vasicek**		-6.69	-26.10	-8.60	-30.03
CIR			-5.67	-8.05	-9.79	CIR			-27.15	-8.58	-30.03
BS				-8.05	-9.79	BS				-7.53	-29.97
VAR					5.89	VAR					-30.04

The critical values are 1.645, 1.96 for Diebold-Mariano test and 1.282, 1.645 for Clark-West test at the 90%, 95% confidence level, respectively.

Table 9

Diebold-Mariano (for non-nested models) and Clark-West tests (for nested models) results for the forecasts generated from five-factor continuous-time models and VAR(1) and AR(1) models. The Clark-West test results are entered in italic font. The models in bold produced the best forecasting results over the period 2 April 2013 to 25 March 2014.

LIBOR5F						Spot5F					
1W	Vasicek	CIR	BS	VAR	AR1	1Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*55.44*	*21.39*	*120.42*	-47.33	-20.04	CKLS	*0.28*	*-0.52*	*-1.34*	-10.13	-10.09
Vasicek		45.27	33.88	-47.31	-19.99	Vasicek		0.92	0.74	-10.13	-10.09
CIR			-64.77	-47.34	-20.04	CIR			0.48	-10.13	-10.09
BS				-47.33	-20.03	BS				-10.13	-10.09
VAR					53.20	VAR					0.49
1M	Vasicek	CIR	BS	VAR	AR1	7Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*97.72*	*58.57*	*26.85*	-25.80	-21.93	CKLS	*-2.48*	*-1.80*	*-3.05*	-8.66	-26.84
Vasicek		6.79	38.09	-25.79	-21.92	Vasicek		2.39	-1.21	-8.66	-26.84
CIR			47.89	-25.79	-21.91	**CIR**			-2.14	-8.66	-26.84
BS				-25.79	-21.93	BS				-8.66	-26.84
VAR					23.20	VAR					-26.76
3M	Vasicek	CIR	BS	VAR	AR1	10Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*69.97*	*-34.22*	*-14.34*	-25.70	-15.43	CKLS	*-2.07*	*-1.35*	*-2.51*	-9.48	-7.53
Vasicek		-3.49	19.94	-25.70	-15.43	Vasicek		1.98	0.29	-9.48	-7.53
CIR			43.81	-25.69	-15.42	**CIR**			-1.67	-9.48	-7.53
BS				-25.69	-15.43	BS				-9.48	-7.53
VAR					30.54	VAR					4.96
6M	Vasicek	CIR	BS	VAR	AR1	15Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*12.74*	*183.22*	*294.43*	-7.49	-9.78	CKLS	*-0.65*	*0.11*	*-1.33*	-10.14	-15.99
Vasicek		-85.69	-145.37	-7.50	-9.79	Vasicek		0.97	1.72	-10.14	-15.99
CIR			-217.11	-7.41	-9.72	CIR			-0.35	-10.14	-15.99
BS				-7.24	-9.60	BS				-10.14	-15.99
VAR					-7.44	VAR					-12.28
12M	Vasicek	CIR	BS	VAR	AR1	25Y	Vasicek	CIR	BS	VAR	AR1
CKLS	*-1.26*	*185.88*	*90.21*	-8.10	-9.78	**CKLS**	*1.72*	*3.31*	*1.11*	-8.69	-30.03
Vasicek		-94.71	-45.48	-8.10	-9.79	Vasicek		-1.60	0.93	-8.69	-30.03
CIR			141.82	-7.99	-9.45	CIR			1.85	-8.69	-30.03
BS				-8.08	-9.70	BS				-8.69	-30.03
VAR					5.19	VAR					-30.05

The critical values are 1.645, 1.96 for Diebold-Mariano test and 1.282, 1.645 for Clark-West test at the 90%, 95% confidence level, respectively.

Table 10

Clark-West Test Results for the Forecasts Generated from Four- and Five-factor Continuous-time Models and the discrete-time VAR(1) model. This table reports the values of the Clark-West tests for pairs of the same model under four and under five-factors. The models in bold produced the best forecasting results over the period 2 April 2013 to 25 March 2014. The critical values for comparing the test values are 1.645, 1.96 and 2.576 at 90%, 95% and 99% confidence level.

LIBOR 4F/5F	CKLS	Vasicek	CIR	BS	VAR
1W	1.15	79.25	26.99	4.06	47.75
1M	31.86	210.30	32.81	138.34	35.25
6M	-9.36	30.86	15.63	10.80	23.86
12M	8.27	11.77	2.99	9.91	4.79
Spot 4F/5F	CKLS	Vasicek	CIR	BS	VAR
1Y	14.10	17.48	10.49	19.33	-1.65
7Y	13.87	12.83	20.86	21.42	1.58
15Y	15.16	13.22	12.27	23.94	0.62
25Y	17.52	17.40	13.45	25.86	0.14

The critical values are 1.282 and 1.645 for Clark-West test at the 90% and 95% confidence level, respectively.