Evaluation of several reconstruction methods of bandlimited signals.

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***Abstract*—In this paper we evaluate several methods of reconstructing signals from finite sets of their samples. A class of band-limited signals is considered. Both, noise-free and noisy cases are studied. The evaluation is performed by extensive simulations where different shapes and bandwidths of the reconstructing filters are examined. We demonstrate that if a fixed number of signal samples are used in the reconstruction, then the signal to noise ratio becomes the main factor limiting the quality of the reconstruction.**

I. INTRODUCTION

The textbook solution to reconstructing baseband signals from their samples is Shannon's formula:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\operatorname{sinc}(tf_S - n)\,, \tag{1}$$

where $f_S = T^{-1}$ is the sampling frequency. In practice only a finite number of signal samples are available and (1) has to be truncated to a finite number of components. An important question arises here how accurate a signal reconstruction is, when obtained from a truncated formula. Specifically, how big the following truncation error is:

$$E(t) = x(t) - \sum_{n=-N}^{N} x(nT)\operatorname{sinc}(tf_S - n)\,, \tag{2}$$

where $2N+1$ is the number of collected samples. Assessing error (2) is generally not a straightforward task since it depends on an infinite number of unknown signal samples.

It has been noticed long ago that the truncated Shannon's formula (TSF) often provides poor reconstruction quality of the processed signal. This observation could be supported by theoretical analysis. For example it can be shown that the truncation error (2) can take arbitrary large values even if it is known that all the missing samples are bounded i.e. $|x(kT)| \le M$. We demonstrate this by estimating the reconstruction error created by neglecting $\{x(nT)\}_{n=N+1}^{\infty}$ samples in (1). Suppose $t = -0.5T$ and the neglected samples are $x(nT) = (-1)^n M$. In this case the reconstruction error is:

$$\sum_{n=N+1}^{\infty} (-1)^n M \operatorname{sinc}(0.5 - n) = \sum_{n=N+1}^{\infty} \frac{M}{\pi(n-0.5)} = \infty$$

Better reconstructions that do not suffer from the above problem can be obtained by using super-Nyquist sampling rates (oversampling) and reconstruction formulas other than (1) e.g. formulas proposed in [4,5,7]. The considered signals do not have spectral components at frequencies higher than $f_{\max} = rf_S/2 \quad (0 < r < 1)$. We define the guard band as $(1-r)f_S/2$. Besides, signal samples are bounded by a maximum value i.e. M which is a reasonable requirement in practical situations. The spectrum $X_d(f)$ of a sampled signal is related to spectrum $X(f)$ of its continuous-time counterpart:

$$X_d(f) = f_S \sum_{k=-\infty}^{+\infty} X(f + kf_S) \tag{3}$$

The continuous-time signal can be reconstructed by filtering the sampled signal through a filter whose frequency response is:

$$S(f) = \begin{cases} T & \text{when } |f| \le f_{\max} \\ 0 & \text{when } |f| \ge f_S - f_{\max} \\ \text{arbitrary} & \text{when } f_{\max} \le |f| \le f_S - f_{\max} \end{cases} \tag{4}$$

The single-sided bandwidth of the reconstruction filter W can therefore vary between $f_{\max}$ and $f_S - f_{\max}$. This observation is further illustrated by Figure 1. We denote the normalized bandwidth of the filter by $L = W/f_{\max}$ where $1 \le L \le (f_S - f_{\max})/f_{\max}$.

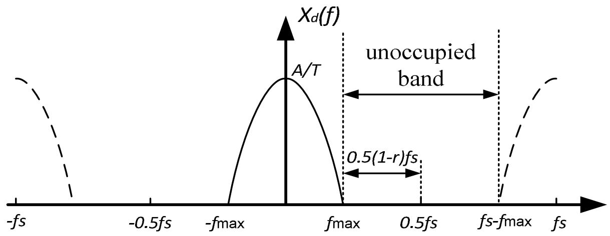

Figure 1, Spectrum of a typical sampled waveform

Oversampling, i.e. presence of an unoccupied band, gives freedom in shaping the reconstruction filter (4) that leads to an infinite number of possibilities in designing different

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reconstructions. However all the relevant formulas reduce to the convolution:

$$x(t) = \sum_{n=-N}^{N} x(nT)s(t-nT) \qquad (5)$$

where $s(t)$ is the impulse response of the filter described by (4). Hence Shannon's formula (1) is only a special case of a huge variety of reconstructions defined by (5). However, there are reconstruction methods e.g. Minimum Energy Reconstruction (MER) [12] that use accustomisation to the layout of the sampling instants and the spectral support of the processed signal. A general formula which represents such methods and includes the ones described by (5) is:

$$x(t) = \sum_{n=-N}^{N} x(nT)s_n(t) \qquad (6)$$

where $s_n(t)$ are carefully selected interpolation functions. Now we redefine the truncation error (2) as:

$$\varepsilon(t) = x(t) - \sum_{n=-N}^{N} x(nT)s_n(t) \qquad (7)$$

There is an extensive literature on estimating the truncation error $\varepsilon(t)$ in an attempt to quantify its upper bound for the considered class of signals [1-8,13,14]. Various analyses have been deployed for this purpose such as complex variables in conjunction with inequalities from theory of functions (e.g. [1, 5, 6]), real variable considerations (e.g. [3]) and bounds on linear systems [2]. [5] and [7] have essentially achieved a better truncation error bound by using reconstruction filters with particular shapes.

It is agreed upon in literature that as the number of the considered samples increases, the bound of the actual error automatically decreases. It is noted that majority of the devised truncation error bounds are strictly upper bounds and are larger than the largest actual error that might be observed during signal reconstruction from a fixed number of samples. This is a clear message that ignoring an infinite number of samples does not necessarily lead to an infinite truncation error during signal reconstitution.

Despite the fact that the truncation error bounds and convergence of classical Shannon Reconstruction has been studied since 1960's [5], there has been little attention paid to the problem of studying/designing reconstruction formulas from noisy samples until 1990's. The first theoretical analysis of Shannon's performance in noisy environment sighted in literature is [9]. It has been noted that the presence of noise causes classical Shannon to break down. Some techniques have been proposed thereafter to tackle the effect of different types of added noise and improve the robustness of Shannon. We note [10], [11] which contain a list of references. However, those papers were dedicated solely to Shannon's method.

In this paper a series of tests are conducted on various reconstruction methods with different filter shapes and widths in an attempt to compare their actual truncation error level in absence as well as presence of noise added to the samples e.g. quantization noise or out-of-band noise. The effect of noise on such methods, excluding Shannon, has not been virtually considered.

II. TESTED RECONSTRUCTION METHODS

In this section we describe the reconstruction methods used in our simulations. The chosen reconstruction filters have been already mentioned in literature e.g. raised cosine [7] and filters obtained from convolving rectangular shapes inspired by the self-truncating reconstruction in [5]. These filters have a single sided bandwidth of W that we vary in our simulations from $f_{\max}$ to $f_S - f_{\max}$.

Rectangular filter has a transfer function $S_r(f) = T$ when $|f| < W$ and $S_r(f) = 0$ when $|f| > W$. The "convolved filters" have a frequency response of:

$$S_m(f) = T(W + f_{\max})\,\mathrm{rect}(\frac{f}{W + f_{\max}}) * \left\{\mathrm{rect}(\frac{mf}{W - f_{\max}})\right\}^m$$

where $\mathrm{rect}(f) = 1$ if $|f| < 0.5$, $\mathrm{rect}(f) = 0$ if $|f| > 0.5$ and m is the number of convolutions. The value m which would achieve the lowest truncation error bound was derived in [5] and is equal to $N(1-r)\pi / e$. By using this value we obtain the self-truncating filter model ($S_{st}(f)$) [5]. Famous Raised Cosine filter and MER method [12] are also tested. The MER method can be stated as:

$$x(t) = 2f_{\max} \sum_{n=-N}^{N} \beta_n \mathrm{sinc}(2f_{\max}(t-nT)) \qquad (8)$$

where β_n parameter can be calculated using the collected signal samples given the fact that MER provides perfect reconstruction at sampling points.

III. SIMULATION

In our simulations we used randomly generated, band-limited signals ($f_{\max} = 500\mathrm{Hz}$) with bounded samples. The presented experiments used 25 samples (*N*=12). The reconstruction error was always measured in the central part of the window:

$$e_n = \sup_{-0.5T \le t \le 0.5T} (|\varepsilon(t)|), \qquad (9)$$

where n denotes the number of reconstruction experiments. We used more than 300 experiments per tested method. To compare the methods we tested, we used the maximum relative reconstruction error defined by:

$$e = 20\log\left[\max_n(e_n) / M\right], \qquad (10)$$

as a measure of the quality of the signal reconstruction where M is the maximum absolute value the samples $x(kT)$ could take.

A parameter, named q, is introduced to represent the oversampling ratio $q = f_S / 2f_{\max}$. White zero mean Gaussian noise is used for examining the effect of noise on the studied methods performance. The variance of the noise was adjusted to provide the selected SNRs. We recall the list of the tested models:

- Shannon
- Rectangular filter
- Selected number of convolved filters namely $S_1(f)$, $S_{21}(f)$ and $S_{st}(f)$
- Raised Cosine $S_{rc}(f)$
- Minimum Energy Reconstruction (MER)

The parameters that have been modified in our tests are: reconstructions filter normalized bandwidth (*L*), oversampling ratio (*q*) and signal to noise ratio (SNR).

A. Noise Free Case

Figure 2 shows the plots of maximum reconstruction error versus filter normalized single sided bandwidth. It is noted from Figure 2 that MER clearly has the best performance with a considerable margin compared to the other tested methods which can be represented by (5).

We also note that in general extending the bandwidth of the filter improves the quality of signal reconstruction. $S_r(f)$ is an exception. Shannon and MER are not affected since they are independent of reconstruction filter normalized bandwidth.

Figure 2 confirms observations made in [5] and [7]. $S_{st}(f)$ with a single sided bandwidth of $f_S - f_{\max}$ provides the best performance for methods represented by (5). Similarly, [11] states that raised cosine filter produces a truncation error lower than that of Shannon's.

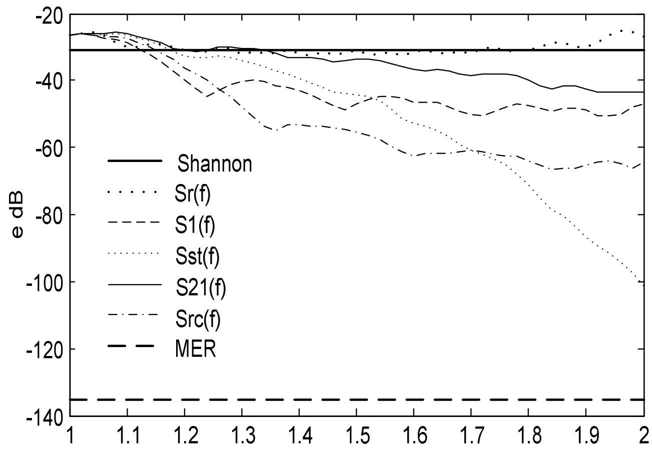

Figure 2, Maximum reconstruction error versus normalized filter bandwidth for noise free case.

Figure 3 shows maximum reconstruction error for various oversampling ratios whilst the filter single sided normalized bandwidth is set to 2 ($W = f_S - f_{\max}$). The maximum error is decreasing as the oversampling ratio increases for all the tested methods. However, MER performance starts deteriorating after *q*=1.7. This is solely due to numerical errors. MER involves solving a set of linear equation. The use of finite precision arithmetics can introduce errors in case of handling badly conditioned set of linear equations. Such errors could be the dominating factor limiting the quality of signal reconstruction. This is a clear message that MER does have the lowest maximum reconstruction error for noise free cases but this comes at the expense of computations which can introduce errors.

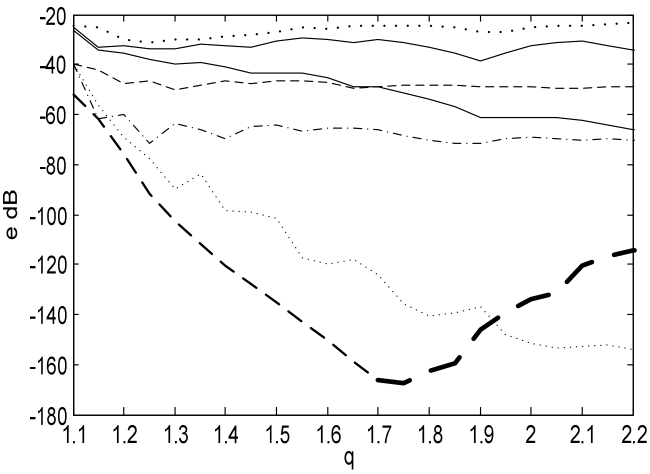

Figure 3, Maximum reconstruction error versus oversampling ratio for noise free cases.

B. Noisy Case

In this experiment the collected samples were contaminated with noise. It can be observed from Figure 4 that MER performance was significantly affected by noise and it ceases to have the lowest maximum reconstruction error as SNR decreases. This is due to the sensitivity of the method and the use of the noisy samples in the approximation of the sought interpolation functions.

Besides, it can be noticed that as SNR drops, the minimum bandwidth of the filter used in (5) which offers the best reconstruction decreases. Exceeding these minimum bandwidths of the filter hardly affects the quality of reconstruction. Third observation is that with decreasing SNR, the shape of the filter or kind of method used becomes less relevant to the quality of the reconstruction – more of them provide the same/better quality as the theoretically best method.

As expected the best achievable quality of reconstruction deteriorates with SNR declining. In the case SNR = 15 dB the difference between the best and the worst reconstruction is insignificant.

C. SNR and Sampling Frequency Changing

The error bounds reported in [1,3,7] clearly indicated that the quality of reconstruction should improve with increasing the oversampling ratio q. However, they considered only noise-free cases. Here we tested the quality of reconstruction for fixed filter normalized frequency of 2, various values of SNR and oversampling ratios. We are only showing three of the tested methods as all methods feature similar trends.

Referring to Figure 5, we note that in general increasing the oversampling ratio q leads to improving the quality of reconstruction. There is however a threshold above which increasing oversampling ratio does not provide any benefits. This threshold varies with SNR. It becomes smaller if SNR decreases. For example, in Shannon's case if SNR<30dB then the threshold oversampling ratio is 1.1. There would be no gain from increasing the sampling rate above this level. Besides Figure 5 shows that for certain q, the lowest maximum reconstruction error is achieved following a threshold SNR above which increasing level of the signal would not improve the quality of the reconstruction process.

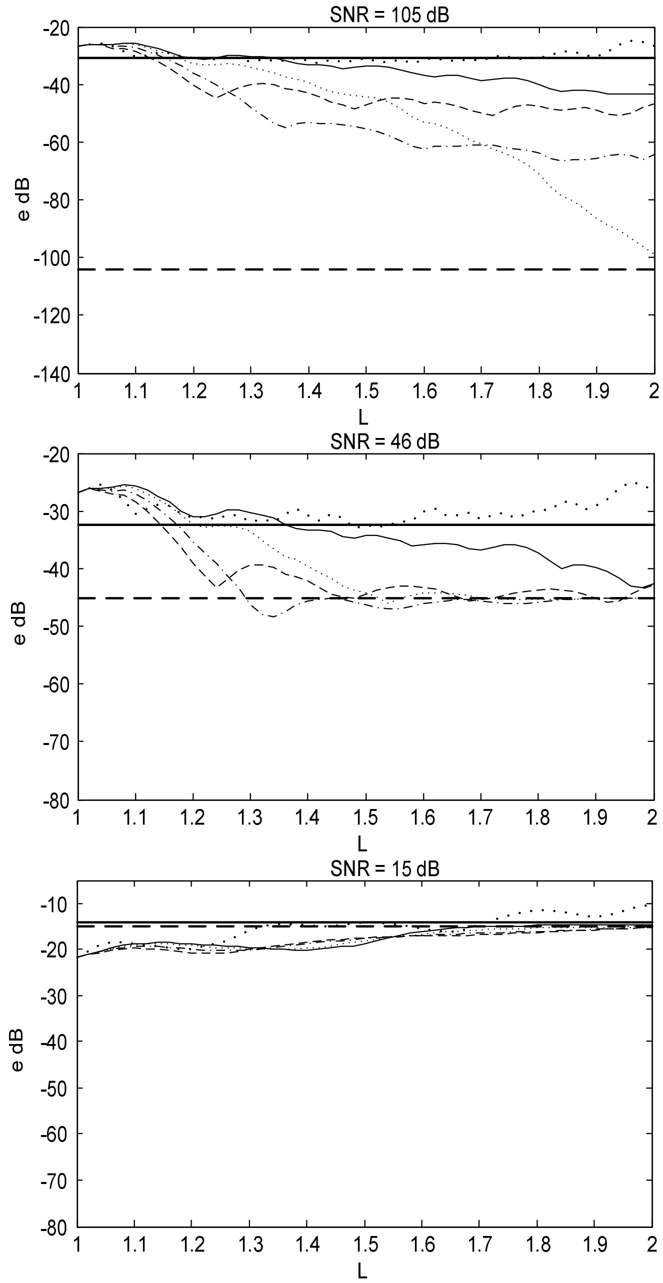

Figure 4, Maximum reconstruction error for various filter normalized bandwidths for noisy cases.

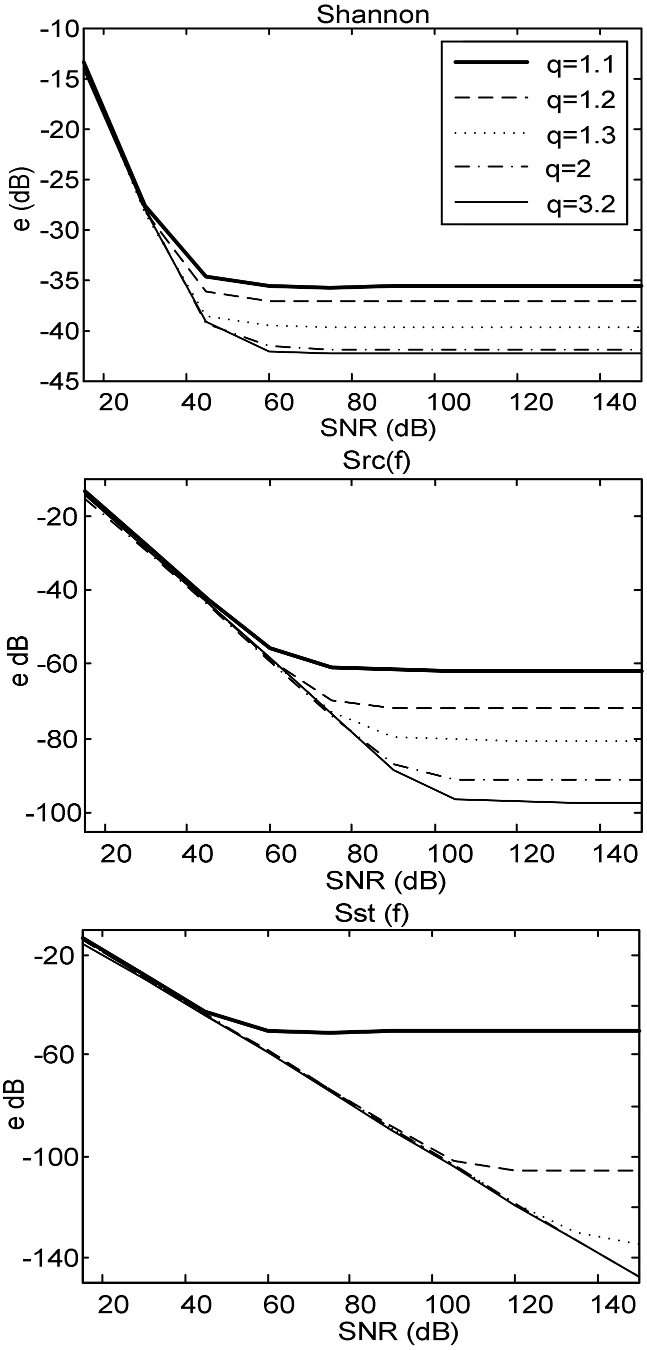

Figure 5, Maximum reconstruction error for selected methods with changing SNR and oversampling ratios.

IV. Conclusion

In this paper we have evaluated a number of methods of reconstructing signals from their samples. Our interest was to gain an insight into the ways of minimizing / controlling reconstruction errors when only a finite number of samples were available. In such cases perfect reconstruction formulas had to be truncated to finite number of components. We have also investigated how such methods behave when the measured samples were contaminated by noise. The analysed signals were random, bounded and band-limited. The samples were taken at the rates exceeding Nyquist.

The considered methods were tested in different environments by changing filter normalized bandwidths, SNR of the sampled signals and oversampling rates. The results we obtained provided a consistent image that allowed us to draw a number of conclusions listed in this paper.

The most important observation was that the SNR is the main factor that limits the quality of reconstruction. This fact is normally ignored in research literature devoted to search for optimal signal reconstruction methods from finite numbers of signal samples. We have also found that as SNR decreases the difference between the quality of different reconstruction methods becomes smaller and the family of "suboptimal" reconstructions, i.e. those that provide performance close to the best achievable, visibly grows. This observation reduces to some extent the necessity of laborious search for signal reconstruction method that minimizes errors. Finally, in the presence of noise, increasing the oversampling ratio does not necessarily lead to more accurate reconstruction. There is a threshold above which negligible benefit is gained from increasing the sampling rate.

The evaluations in this paper were done by simulation. In order to strengthen our confidence that research outcomes reported here represent typical rather than special cases, we ran a number of consistency tests. This research work gathered a large amount of experimental evidence on the quality and usefulness of various reconstruction methods. This evidence will be now used to inform our future, more theoretical analyses that aim at formulation quantitative recommendations on how to practically perform signal reconstruction and modeling in noisy environments.

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